The ABC of Absorbing Boundary Conditions

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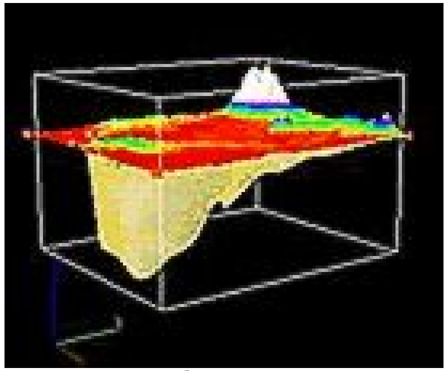
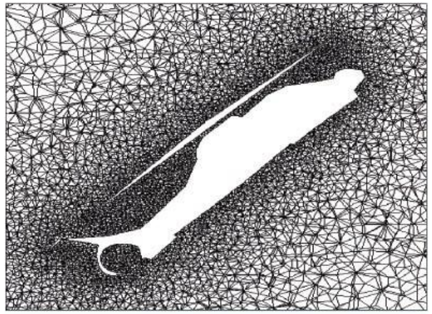
Waves in Unbounded Media

Applications:

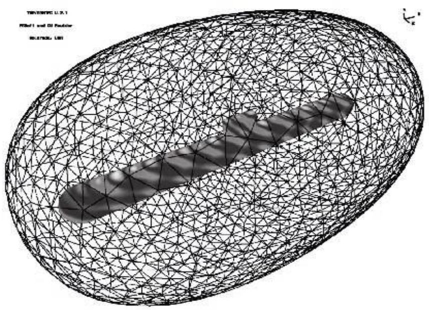
- Geophysics
- Acoustics
- Aerodynamics
- Oceanography
- Meteorology
- Electromagnetics
- and more ……..



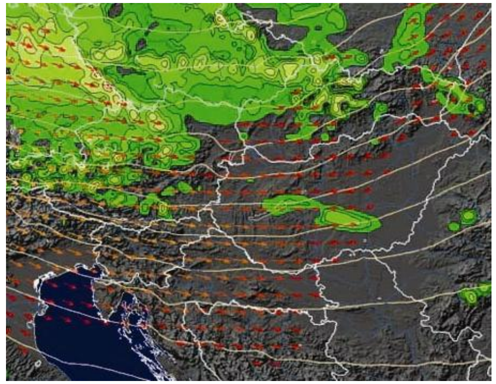
J. Tromp, CalTech



Finite element model of the flow/waves around a rotorcraft (Tayfun E. Tezduyar and Brian Matheis)

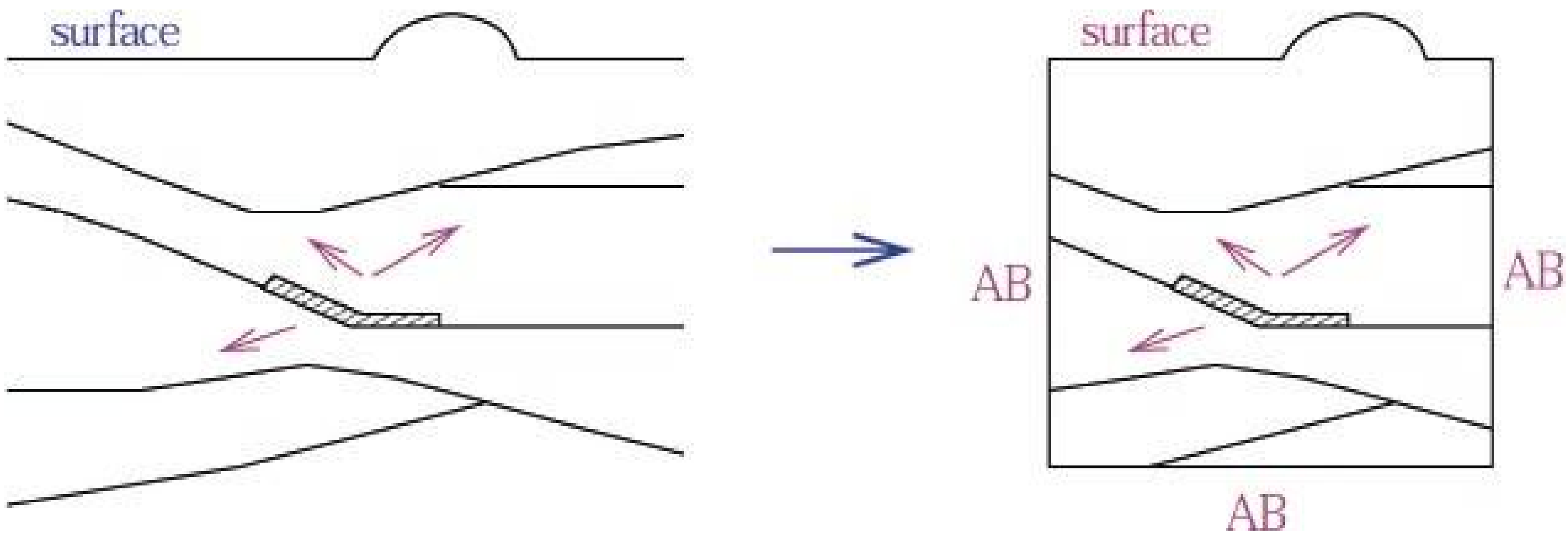


C. Farhat et al.

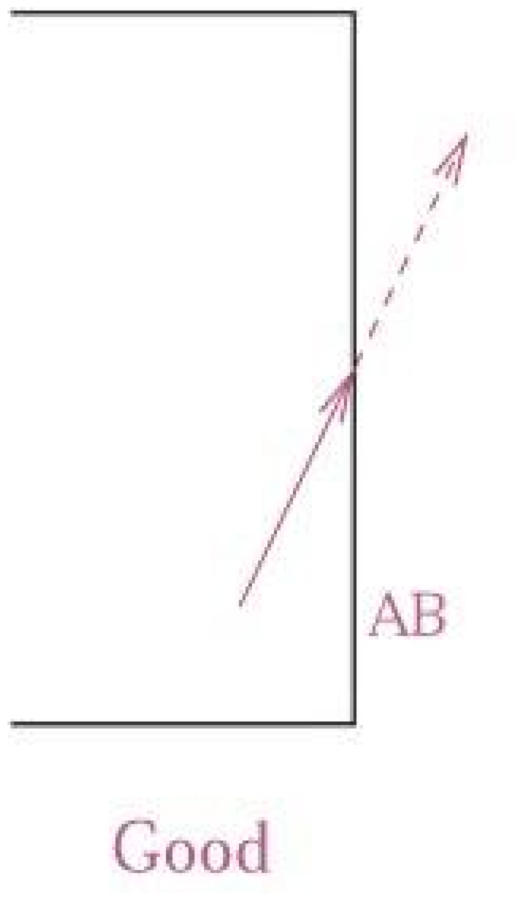


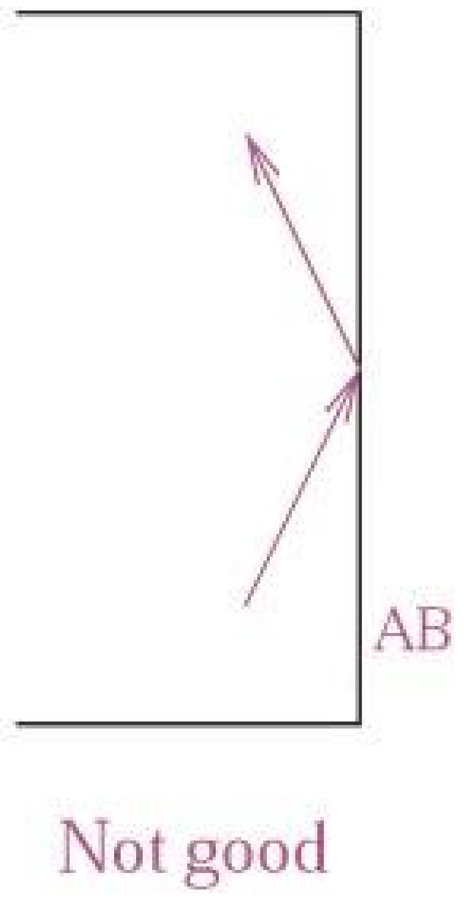
Hungarian Meteorological Society

Artificial / Absorbing Boundaries



Artificial / Absorbing Boundaries





The 1D Case: Basics

The wave equation in 1D: $U_{tt} = C^2 U_{xx}$

C = wave speed

Time-harmonic waves: $U(x,t) = u(x)e^{-i\omega t}$

ω = wave frequency

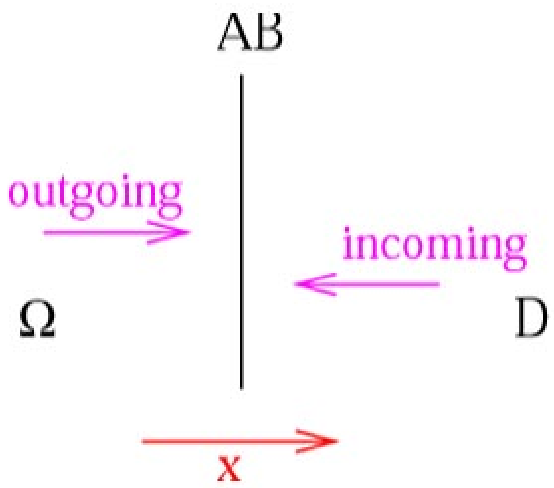
$\Rightarrow$ Reduced wave eq. (Helmholtz eq.):

$$u_{xx} + k^2 u = 0$$

$k = \omega / C$ = wave number

Outgoing wave: $\exp(ikx)$, $G(x - Ct)$

Incoming wave: $\exp(-ikx)$, $F(x + Ct)$



The Sommerfeld Radiation Condition

Sommerfeld [1964]: A mathematical condition which states that at infinity all waves are **outgoing**

The Sommerfeld Radiation Condition – 1D

Time-harmonic case:

$$\boxed{u_x - iku = 0}$$

Check:

$$u = \underbrace{c_1 \exp(ikx)}_{\text{outgoing}} + \underbrace{c_2 \exp(-ikx)}_{\text{incoming}}$$

$$u_x - iku = 0 \Rightarrow$$

$$c_1 ik \exp(ikx) - c_2 ik \exp(ikx) - ik[c_1 \exp(ikx) + c_2 \exp(-ikx)] = 0 \Rightarrow$$

$$-2ikc_2 \exp(-ikx) = 0 \Rightarrow$$

$$c_2 = 0 \Rightarrow$$

No incoming wave!

The Sommerfeld Radiation Condition – 1D

Time-dependent case:

$$\boxed{u_x + \frac{1}{C} u_t = 0}$$

Check:

$$u(x,t) = \underbrace{F(x+Ct)}_{\text{incoming}} + \underbrace{G(x-Ct)}_{\text{outgoing}}$$

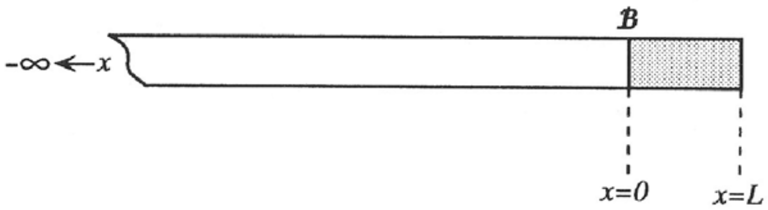
$$u_x + \frac{1}{C} u_t = 0 \Rightarrow$$

$$F'(x+Ct) + G'(x-Ct) + \frac{1}{C}[CF'(x+Ct) - CG'(x-Ct)] = 0 \Rightarrow$$

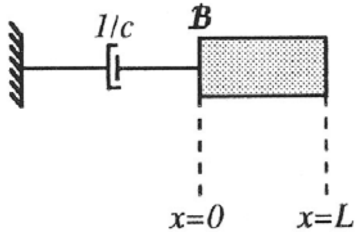
$$2F'(x+Ct) = 0 \Rightarrow$$

No incoming wave!

The dashpot effect:

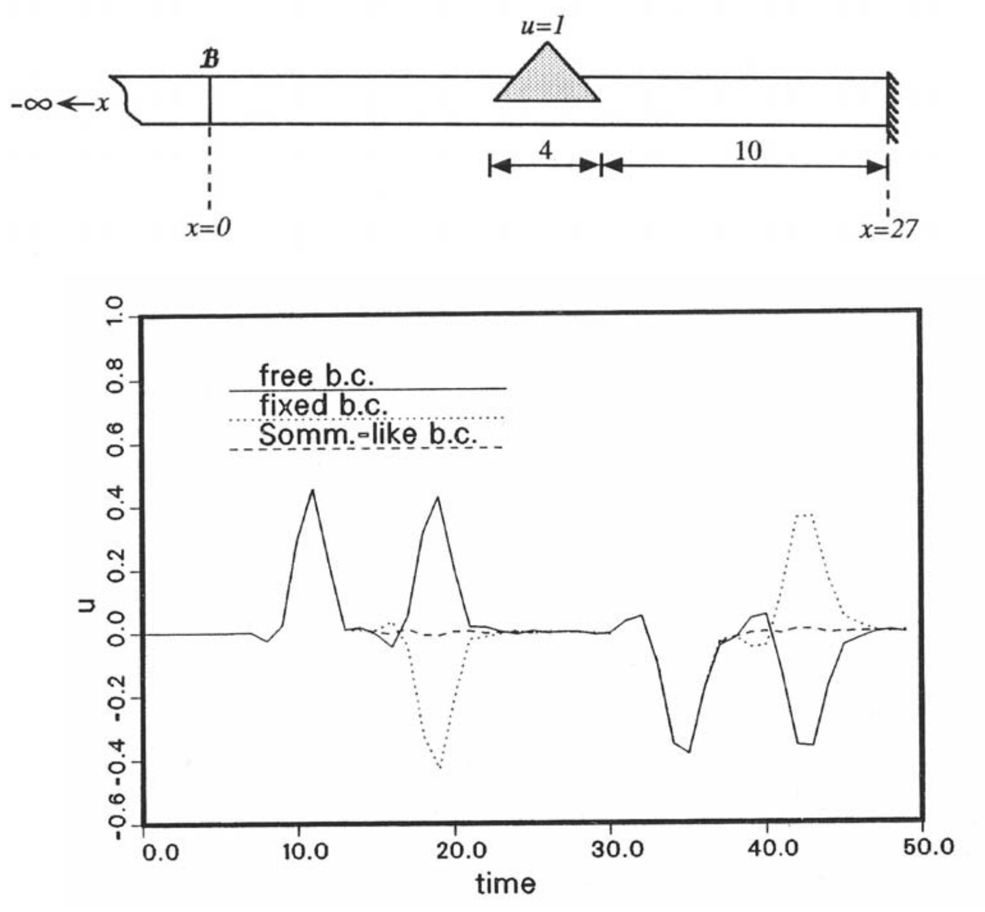


$$\Leftrightarrow$$



The Sommerfeld Radiation Condition – 1D

Demonstration:



The Sommerfeld Radiation Condition - General

The Sommerfeld Condition [1964] states that

there are no incoming waves from infinity.

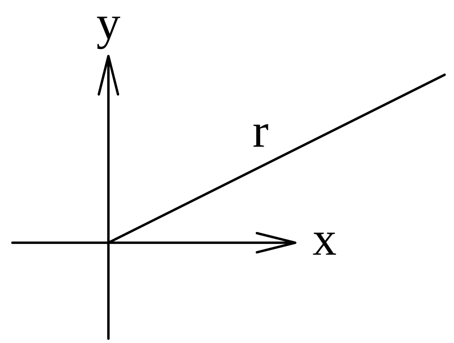
In D dimensions:

Time-harmonic case $$\lim_{r\to\infty} r^{(D-1)/2}(u_r - iku) = 0$$

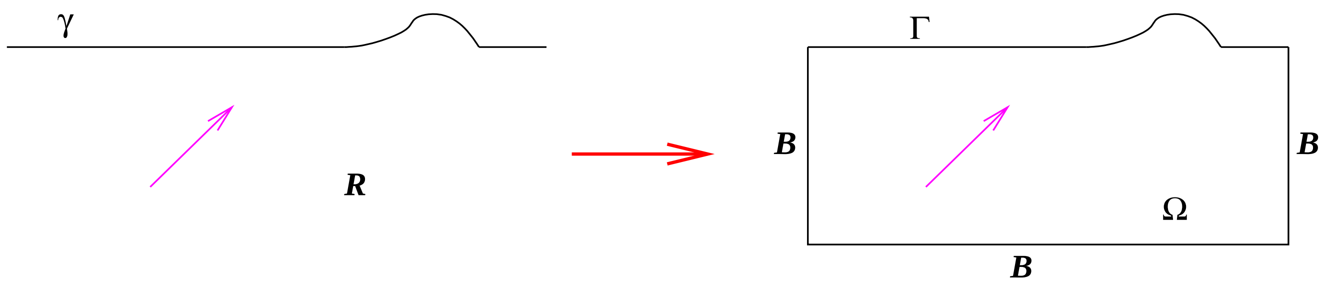
Time-dependent case $$\lim_{r\to\infty} r^{(D-1)/2}(u_r + \frac{1}{C}u_t) = 0$$

E.g., in 2D, time dependent case:

$$\lim_{r\to\infty} \sqrt{r}\,(u_r + \frac{1}{C}u_t) = 0$$



Absorbing Boundary Conditions



Time harmonic

$$\nabla^2 u + k^2 u = 0 \quad \text{in} \quad R \qquad\qquad \nabla^2 u + k^2 u = 0 \quad \text{in} \quad \Omega$$

$$\alpha \frac{\partial u}{\partial \nu} + \beta u = g \quad \text{on} \quad \gamma \qquad\qquad \alpha \frac{\partial u}{\partial \nu} + \beta u = g \quad \text{on} \quad \Gamma$$

$$\lim_{r \to \infty} r^{(D-1)/2} \left(\frac{\partial u}{\partial r} - iku \right) = 0$$

Some BC on B

Time dependent

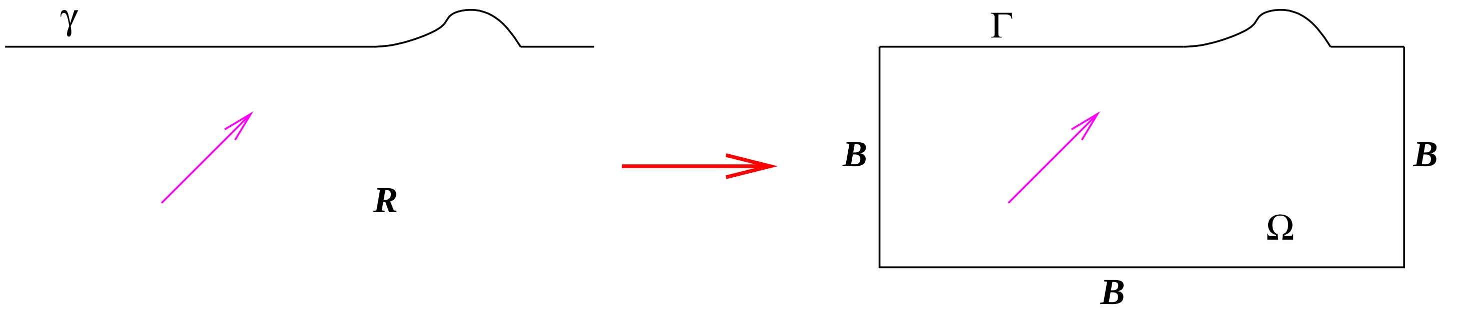
$$\frac{\partial^2 u}{\partial t^2} = C^2 \nabla^2 u \quad \text{in} \quad R \qquad\qquad \frac{\partial^2 u}{\partial t^2} = C^2 \nabla^2 u \quad \text{in} \quad \Omega$$

$$\alpha \frac{\partial u}{\partial \nu} + \beta u = g \quad \text{on} \quad \gamma \qquad\qquad \alpha \frac{\partial u}{\partial \nu} + \beta u = g \quad \text{on} \quad \Gamma$$

$$\lim_{r \to \infty} r^{(D-1)/2} \left(\frac{\partial u}{\partial r} + \frac{1}{C} \frac{\partial u}{\partial t} \right) = 0$$

Some BC on B

Milestones: The Simplest ABC



Till the late 70's:

The Sommerfeld-like BC

Time harmonic: $$\frac{\partial u}{\partial n} - iku = 0 \quad \text{on} \quad B$$

Time dependent: $$\frac{\partial u}{\partial n} + \frac{1}{C}\frac{\partial u}{\partial t} = 0 \quad \text{on} \quad B$$

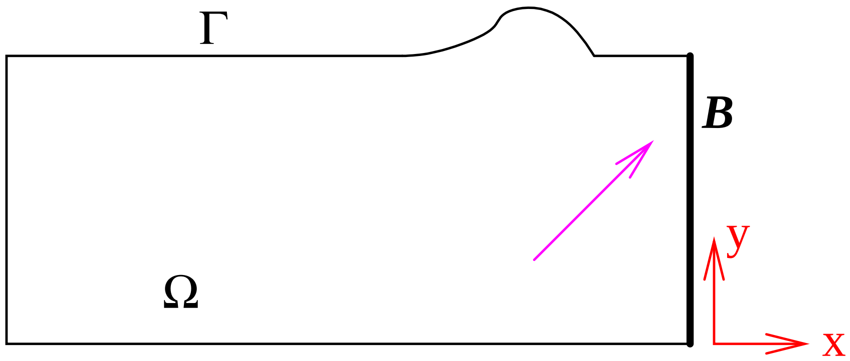
Remark: In geophysics, take $C = C_L$ for u = normal displacement,

and $C = C_T$ for u = tangential displacement.

Milstones: Lysmer & Kuhlemeyer [1969]

$$T_x = a\rho C_L \frac{\partial u_x}{\partial t}$$

$$T_y = b\rho C_T \frac{\partial u_y}{\partial t}$$

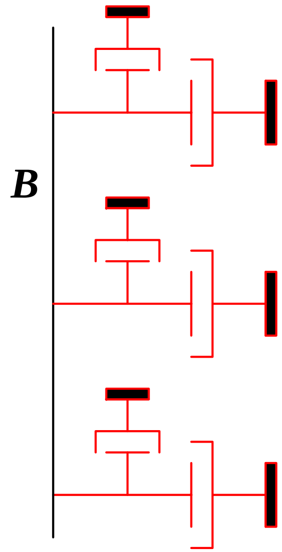


ρ = mass density

C_L = P-wave speed

C_T = S-wave speed

a,b = non-dimensional parameters (usually $a=b=1$)



The dashpot model

Demonstration of Difficulty

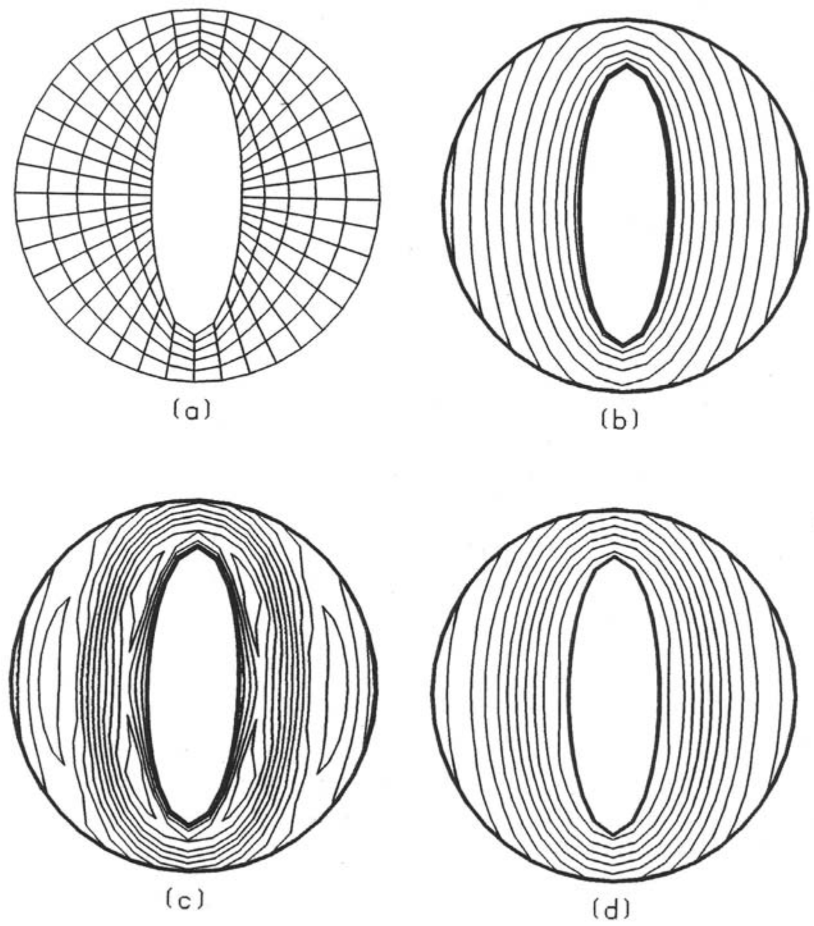
The elliptic obstacle problem:

(a) The finite element mesh

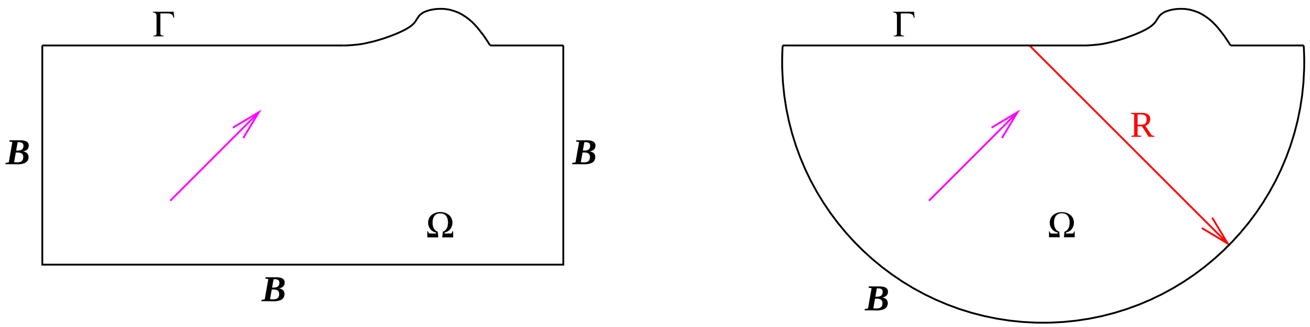
(b) Exact solution

(c) Solution obtained by using the Sommerfeld-like ABC

(d) Solution obtained by using a 1st order ABC



Milestones: Improved ABCs



Late 70's – mid 80's:

Improved Absorbing Boundary Conditions (ABCs)

Other names: Non-reflecting, Radiating, Open, Silent, Transmitting, Transparent, Free-space, Pulled-back, One-way BCs…

Local ABCs:

Engquist & Majda (1977), Bayliss & Turkel (1980), Kriegsmann et al. (1980), Feng (1983), Higdon (1986), Kallivokas & Bielak (1991)…

The Bayliss-Turkel ABCs for a spherical B:

$$B_J u \equiv \left(\prod_{j=1}^{J} \left(\frac{1}{C} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{2j-1}{r} \right) \right) u = 0 \quad \text{on} \quad B$$

Implementation: up to **2nd** order!

Milestones: Variable-Speed ABC

In meteorology, since mid 70's:

Using a Sommerfeld-like ABC with variable C:

$$\frac{\partial u}{\partial n} + \frac{1}{C(\mathbf{x},t)} \frac{\partial u}{\partial t} = 0 \quad \text{on} \quad B$$

$C(\mathbf{x},t)$ is estimated adaptively on B.

Orlanski, 1976; Miller & Thorpe, 1981 $\rightarrow$ COAMPS

Computational Boundary Treatment

Is this a hard problem?

Computational Boundary Scheme, Desired Properties:

- Sufficiently general
- Accurate (and robust)
- Stable
- Efficient
- Easy to implement and compatible with interior scheme

The quest continues…

Remark: The geophysical ABC problem is especially complicated due to

- Heterogeneity of medium
- Half-space geometry
- Various types of waves (P, S, Love, Rayleigh,…)

C.J. Randall [Geophysics, 1988]

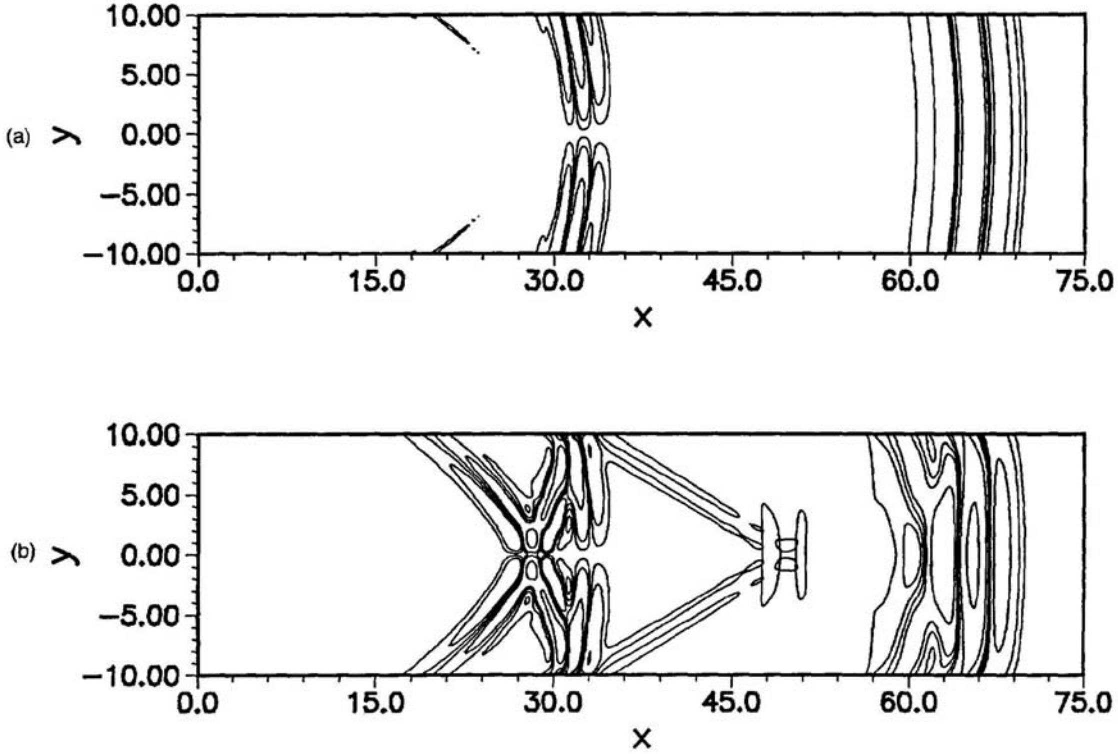


FIG. 5. Contours of $|\nabla \cdot \mathbf{u}|$ and $|\nabla \times \mathbf{u}|$ for 2-D finite-difference calculations with a horizontal point force at the origin. The lowest contour level is 2 percent of the maximum amplitude. (a) New absorbing boundary condition along the upper and lower boundaries. (b) Second-order paraxial absorbing boundary condition along the upper and lower boundaries.

R.A. Renaut & J. Petersen [Geophysics, 1989]

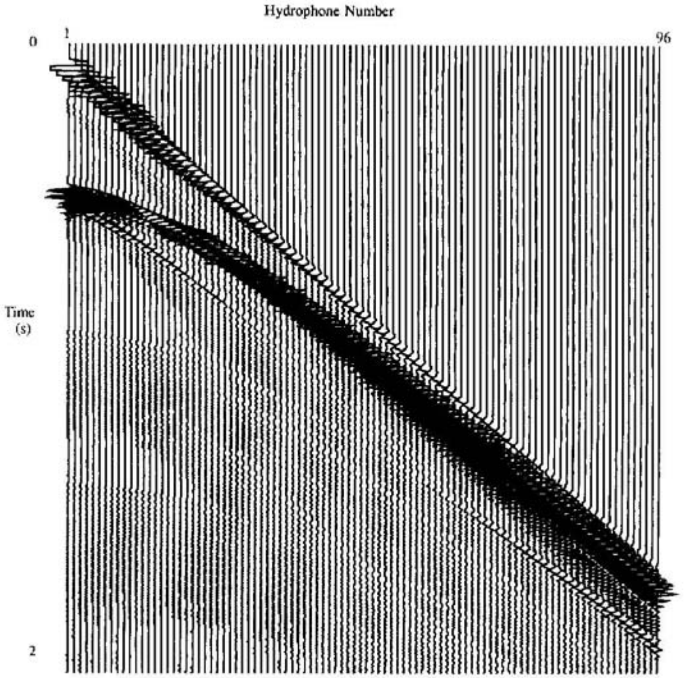


FIG. 8. Seismic record obtained with Padé approximation. A grid size of 2.5 m and a time step of 0.5 ms were used.

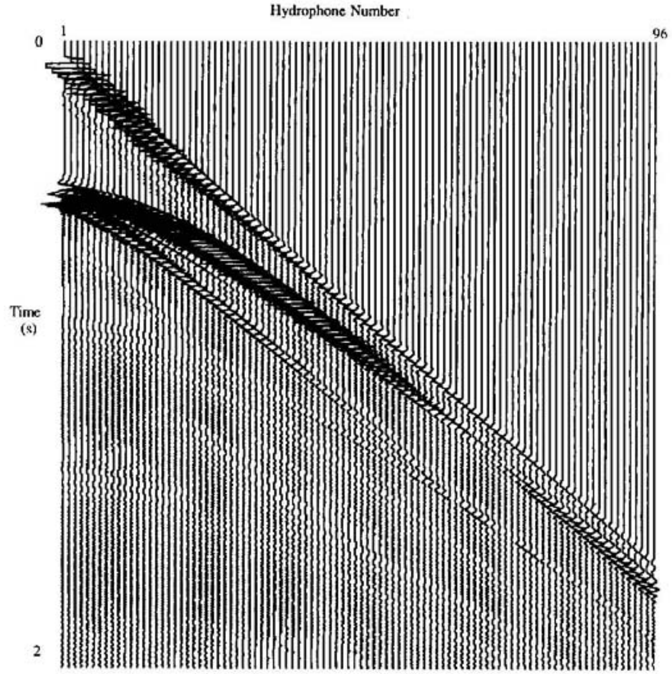
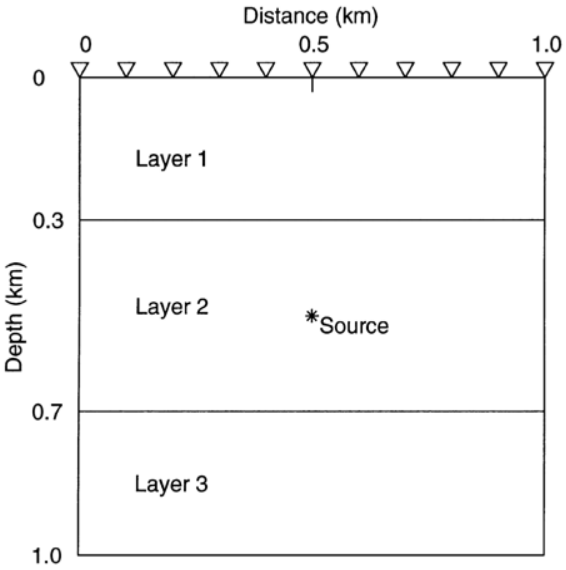
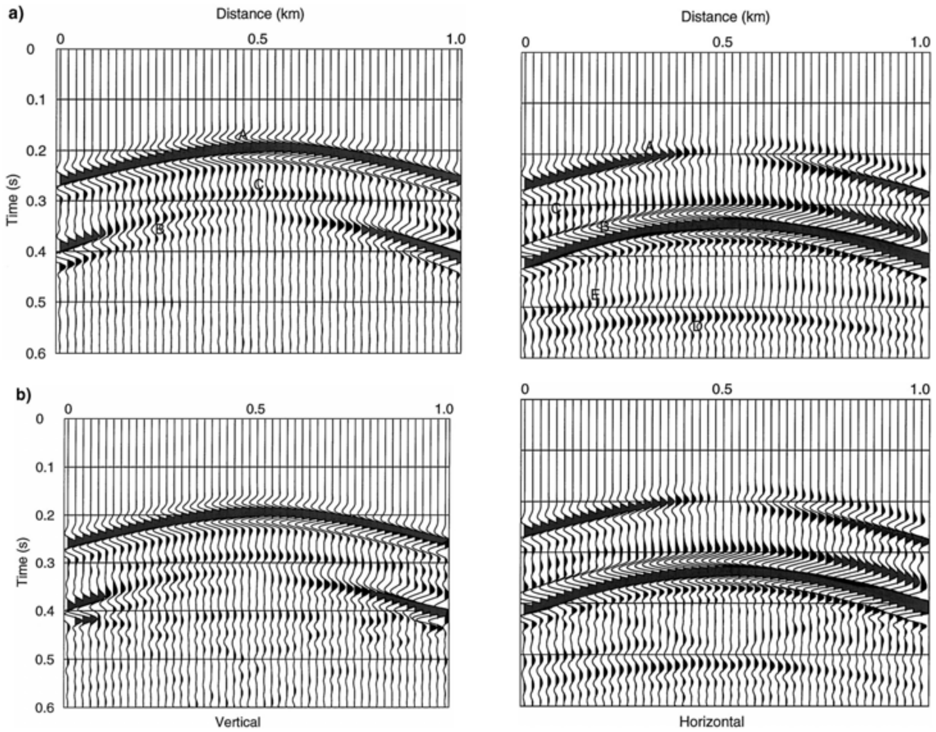


FIG. 10. Seismic record obtained with Chebychev approximation. A grid size of 2.5 m and a time step of 0.5 ms were used.

J. Zhu [Geophysics, 1999]



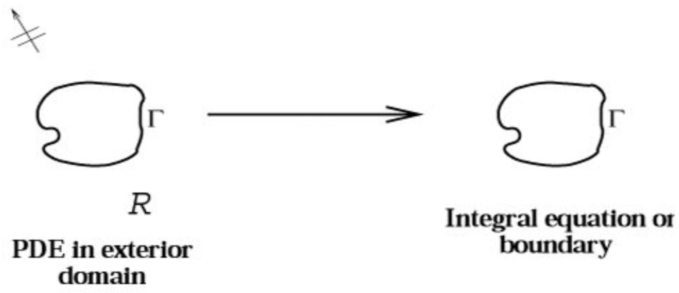


Milestones: BEM

Late 70’s – mid 80’s:

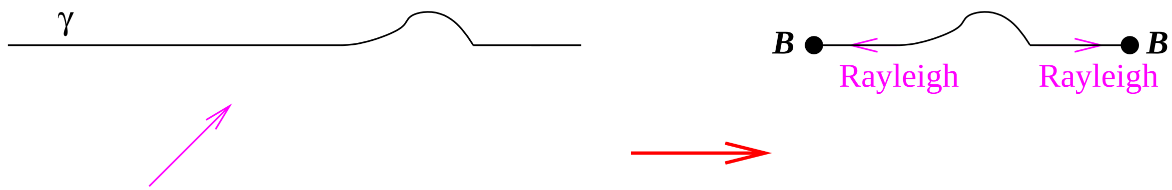
Boundary Integral / Boundary Element Methods

(Cruse & Rizzo and many others)



Associated difficulties: uniqueness, singularity of integral

Recent application to half-space: Arias & Achenbach [2004]



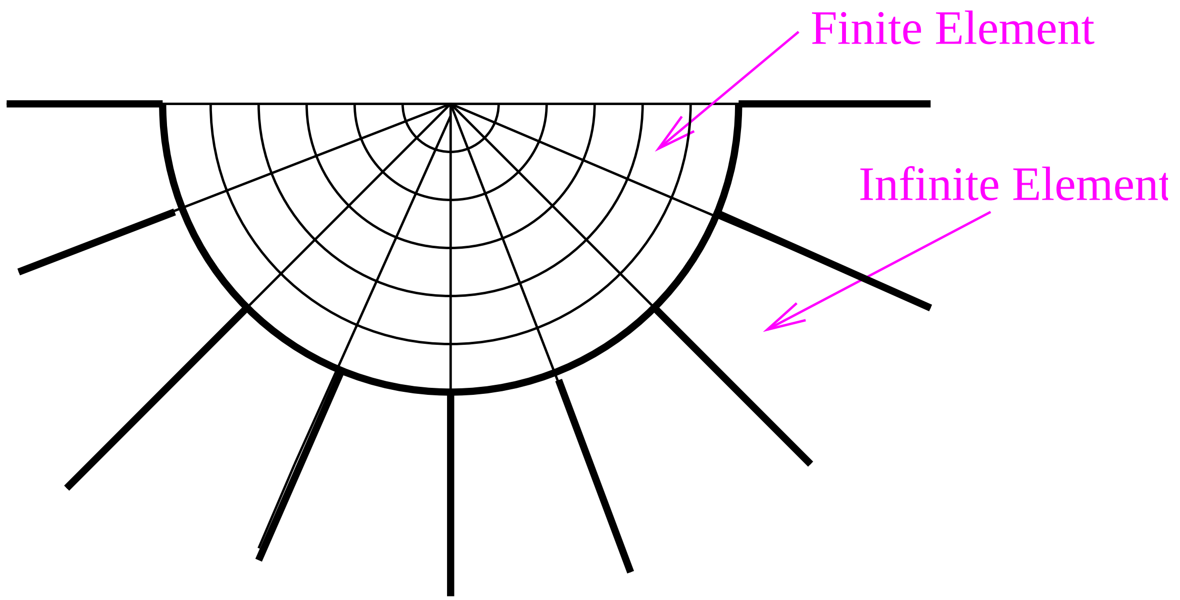
Remark: Applicable only to homogeneous media!

Milestones: Infinite Elements

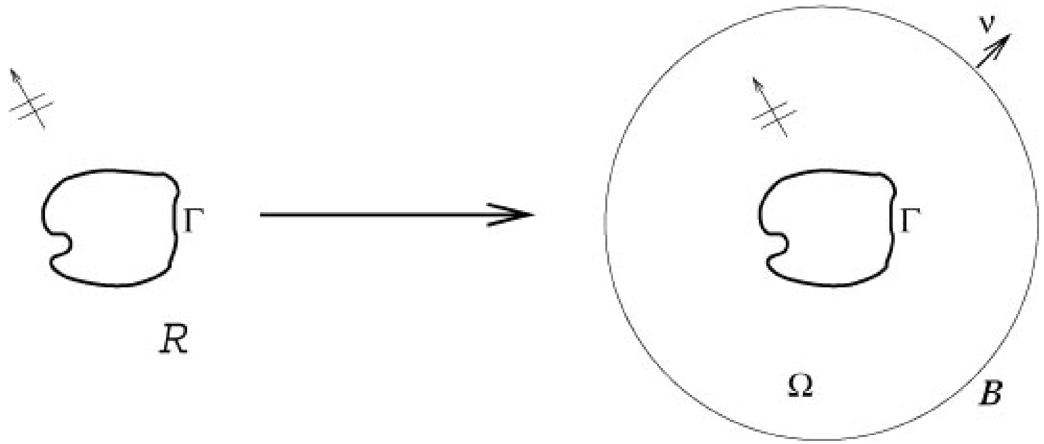
Late 70's – mid 80's:

Infinite Elements

(Bettess and Zienkiewicz)



Milestones: Exact ABCs



Late 80's – mid 90's: Exact Nonlocal ABCs

(Hagstrom & H.B. Keller, J.B. Keller & Givoli, Ting & Miksis,

Lenoir & Tounsi, Tsynkov…)

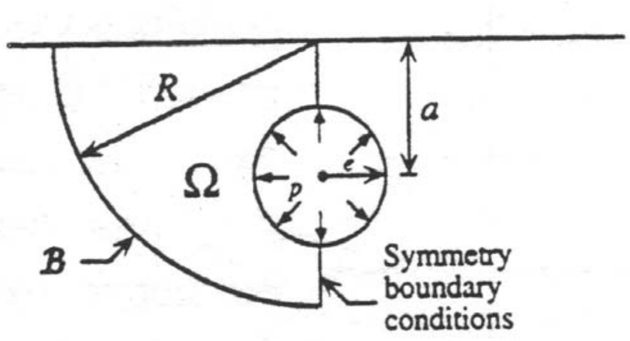
Dirichlet-to-Neumann (DtN) BC for an elliptic problem:

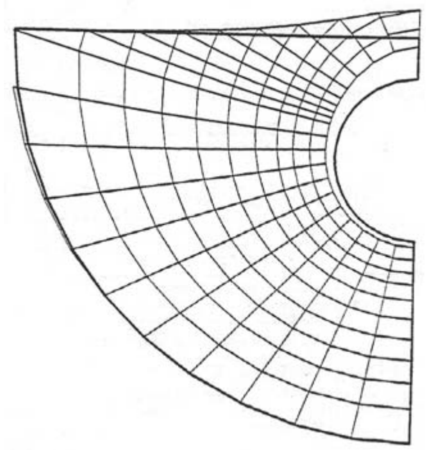
$$-\frac{\partial u}{\partial \nu} = M_J u \equiv \sum_{j=0}^{J}{}' \int_B m_j(\mathbf{x},\mathbf{x}')u(\mathbf{x}')d\mathbf{x}' \quad \text{on} \quad B$$

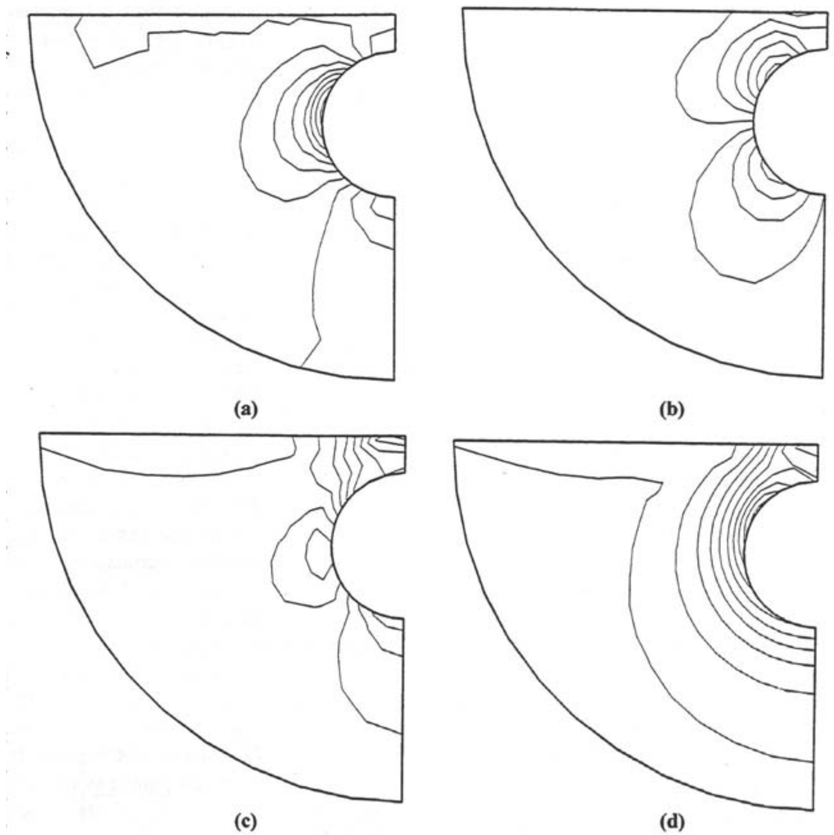
For the Helmholtz exterior problem in 2D, B = circle:

$$m_j(\theta,\theta') = -\frac{k}{\pi R}\frac{H_j^{(1)}{}'(kR)}{H_j^{(1)}(kR)}\cos j(\theta-\theta')$$

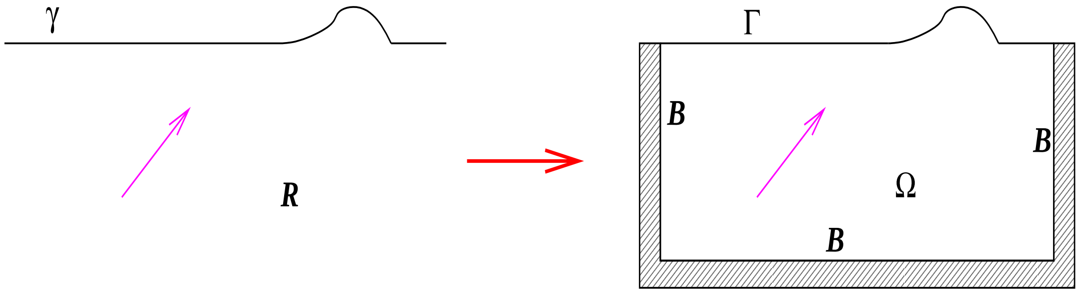
Exact ABC for Elasticity in Half-Space







Milestones: Damping Layers



The 80's:

Damping / Sponge Layers

(Israeli & Orszag, Cerjan et al., Sochacki et al., Kosloff & Kosloff, Karni)

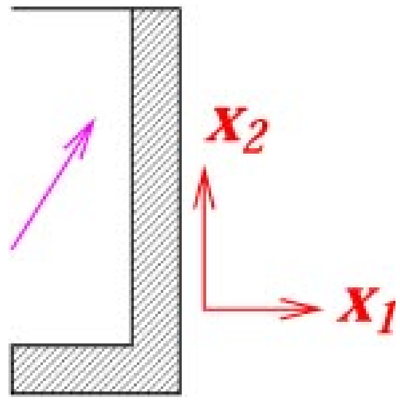
Israeli & Orszag's layer**:**

In the interior:

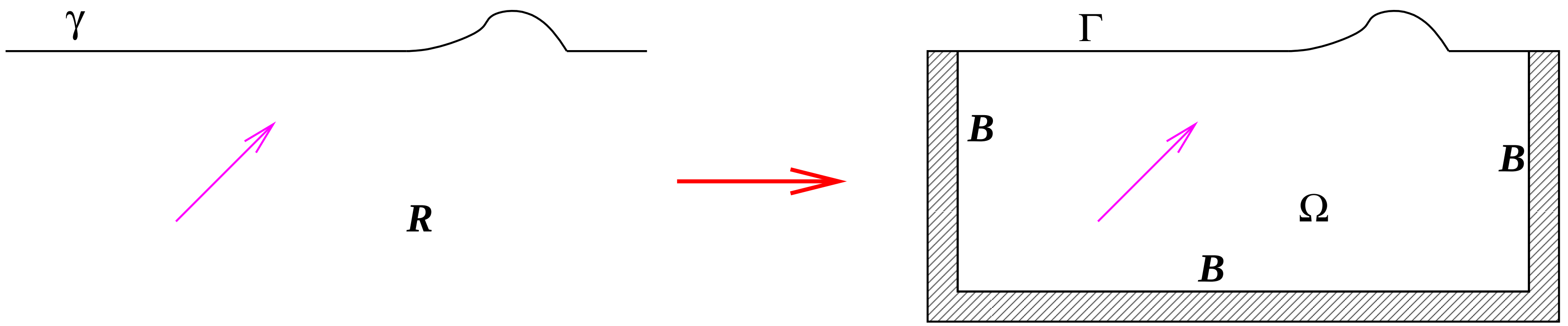
$$\frac{\partial^2 u}{\partial t^2} = C^2 \nabla^2 u$$

In the layer:

$$\frac{\partial^2 u}{\partial t^2} = C^2 \nabla^2 u - \nu(x_1)\left(\frac{\partial u}{\partial x_1} + \frac{1}{C}\frac{\partial u}{\partial t}\right)$$



Milestones: PML



Since mid 90's:

Perfectly Matched Layer (PML)

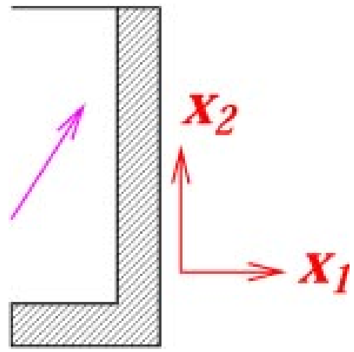
(Bérenger; later many others)

- Zero reflection at the interface B for <u>any</u> plane wave
- Exponential damping inside the layer

PML: Mathematics in a Simple Case

Wave equation:

$$\nabla^2 u - \frac{1}{C^2}\ddot{u} = 0$$

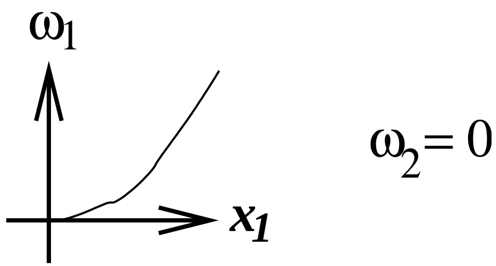


Writing as 1st-order system:

$$\dot{\boldsymbol{v}} = -\nabla u$$

$$\dot{u} = -C^2 \nabla \cdot \boldsymbol{v}$$

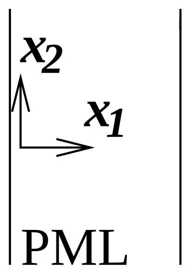
Laplace transform in time:



$$-i\omega \hat{\boldsymbol{v}} = -\nabla \hat{u}$$

$$-i\omega \hat{u} = -C^2 \nabla \cdot \hat{\boldsymbol{v}}$$

Inside PML, coordinate stretching:



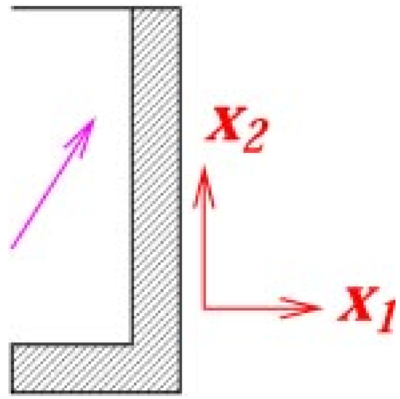
$$\nabla \rightarrow \nabla_s$$

$$\nabla_s = \sum_{j=1}^{3} \boldsymbol{e}_j \frac{1}{S_j} \frac{\partial}{\partial x_j}$$

$$S_j = 1 + i \frac{\omega_j}{\omega}$$

$$\omega_j = \text{PML damping parameter}$$

PML: Mathematics (Contd.)



Thus, in the PML:

$$-i\omega\hat{\boldsymbol{v}} = -\sum_{j=1}^{3} \boldsymbol{e}_j \frac{1}{S_j}\frac{\partial \hat{u}}{\partial x_j}$$

$$-i\omega\hat{u} = -C^2 \sum_{j=1}^{3} \frac{1}{S_j}\frac{\partial \hat{v}_j}{\partial x_j}$$

Splitting the equations:

$$-i\omega\hat{v}_j = -\frac{1}{S_j}\frac{\partial \hat{u}}{\partial x_j}$$

$$-i\omega\hat{u}^{(j)} = -C^2 \frac{1}{S_j}\frac{\partial \hat{v}_j}{\partial x_j}$$

$$\hat{u} = \sum_{j=1}^{3} \hat{u}^{(j)}$$

Substituting $S_j = 1 + i\omega_j / \omega$:

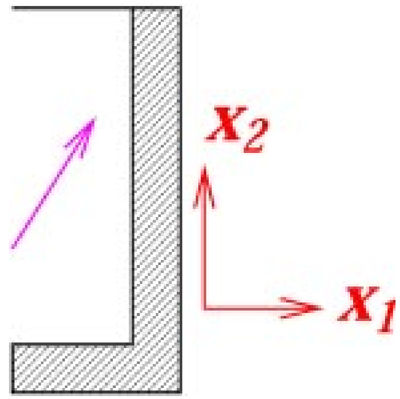
$$(-i\omega + \omega_j)\hat{v}_j = -\frac{\partial \hat{u}}{\partial x_j}$$

$$(-i\omega + \omega_j)\hat{u}^{(j)} = -C^2 \frac{\partial \hat{v}_i}{\partial x_j}$$

These are six equations with six unknowns

PML: Mathematics (Contd.)

Inverse Laplace Transform:



$$\dot{v}_j + \omega_j v_j = -\frac{\partial u}{\partial x_j}$$

$$\dot{u}^{(j)} + \omega_j u^{(j)} = -C^2 \frac{\partial v_j}{\partial x_j}$$

$$u = \sum_{j=1}^{3} u^{(j)}$$

PML equations

(time-dependent)

In the **time-harmonic case**, can obtain a single equation:

$$\sum_{j=1}^{3} \mu_j \frac{\partial^2 u}{\partial x_j^2} + u = 0$$

$$\mu_j = -\frac{1}{(\omega_j - i\omega)^2}$$

Like an anisotropic Helmholtz equation.

PML – Pros and Cons

Pros:

- On the continuous level – <u>perfect</u>, for all waves
- Performs extremely well for high-frequency waves

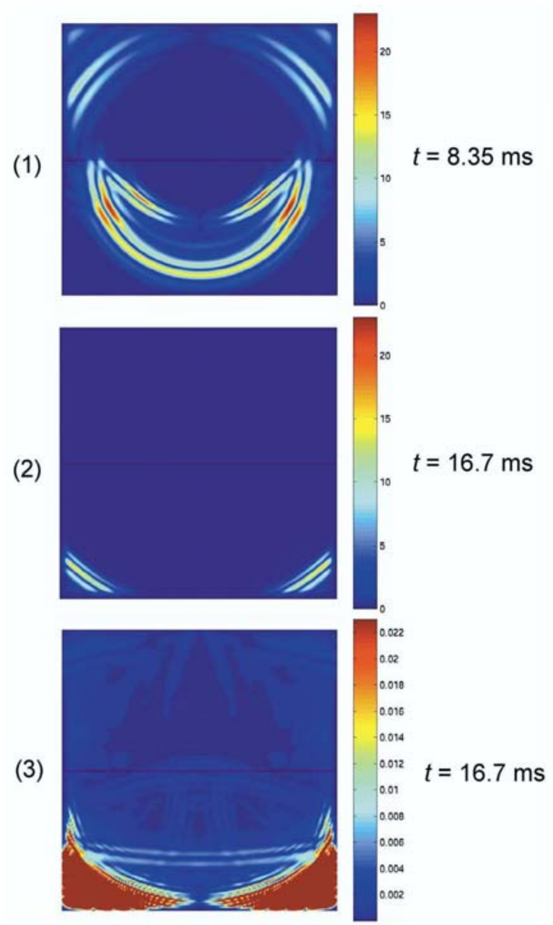
Cons:

- Non-perfect on the discrete level
- Sensitive to choice of parameters ω_j
- Weak-stability difficulties
- No simple sense of convergence
- Performs poorly for $ka<<1$ (Geers 1998)
- Difficulty in treating evanescent modes
- Analysis in the elastic case: not well understood

Still under intensive research

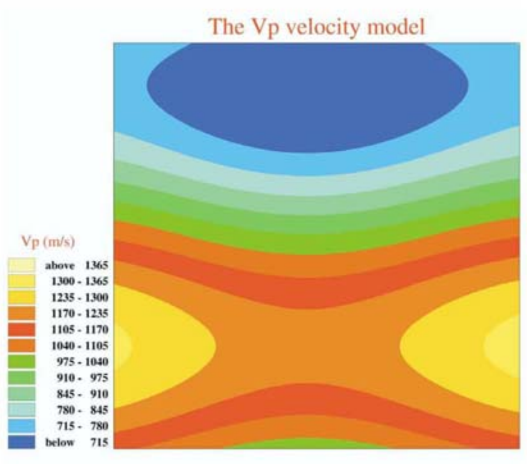
F. Collino & C. Tsogka [Geophysics, 2001]

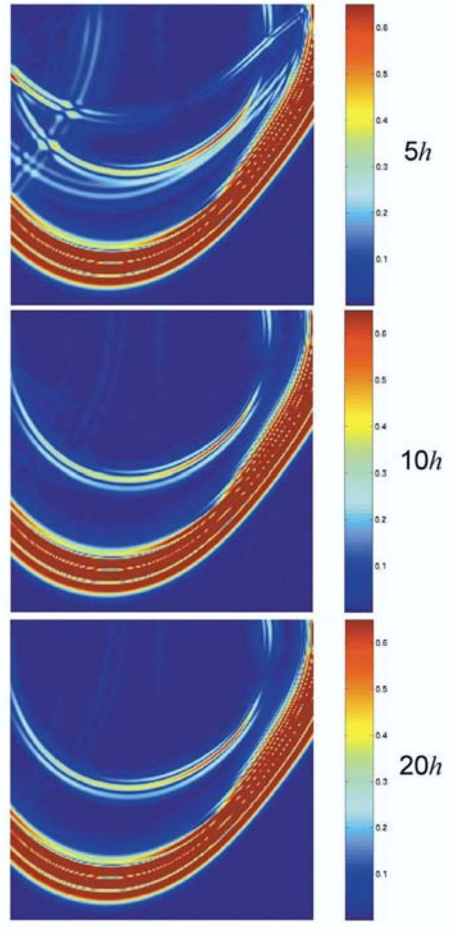
Two-material medium, PML



F. Collino & C. Tsogka [Geophysics, 2001]

Heterogenous medium, PMLs with various thicknesses



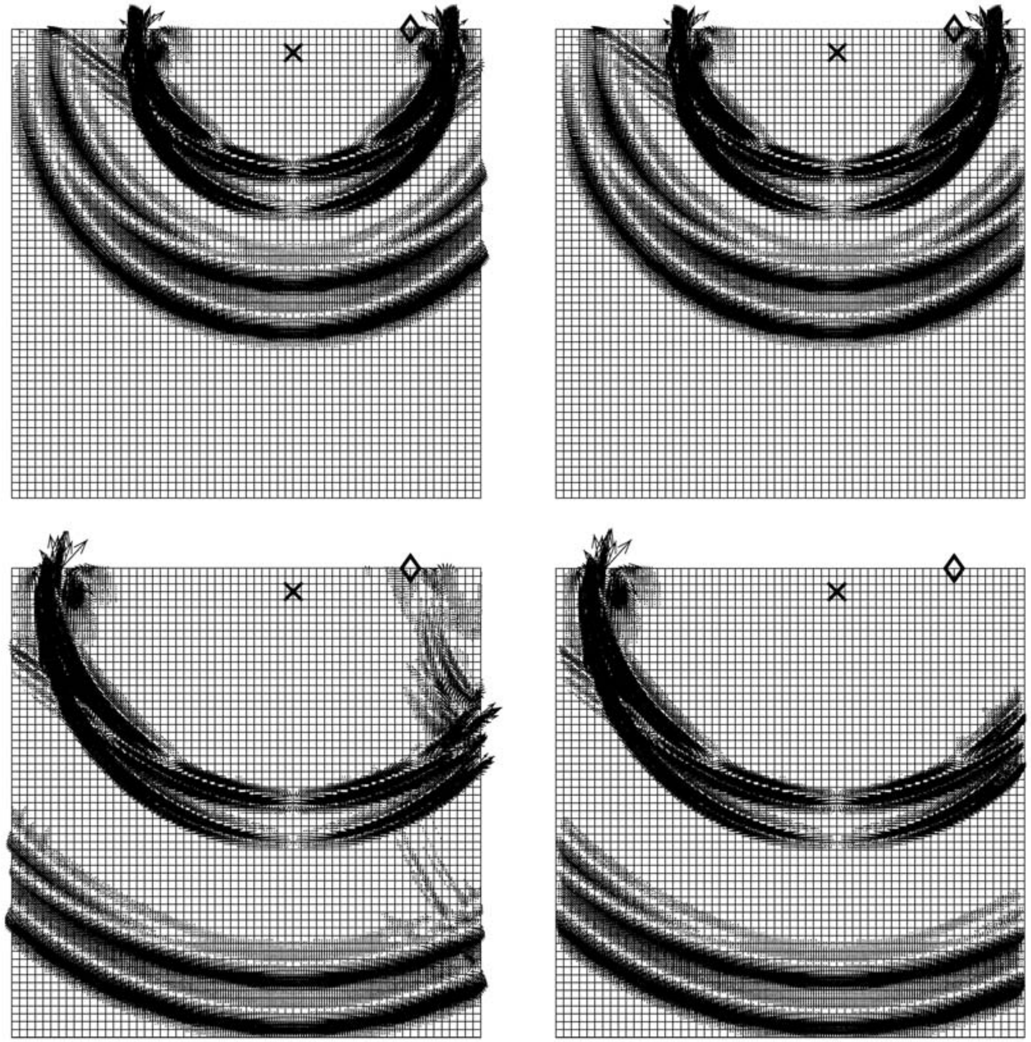


D. Komatitsch & J. Tromp [Geophys. J. Int., 2003]

1st-order ABC | **PML**

t = 10 ms

t = 15 ms

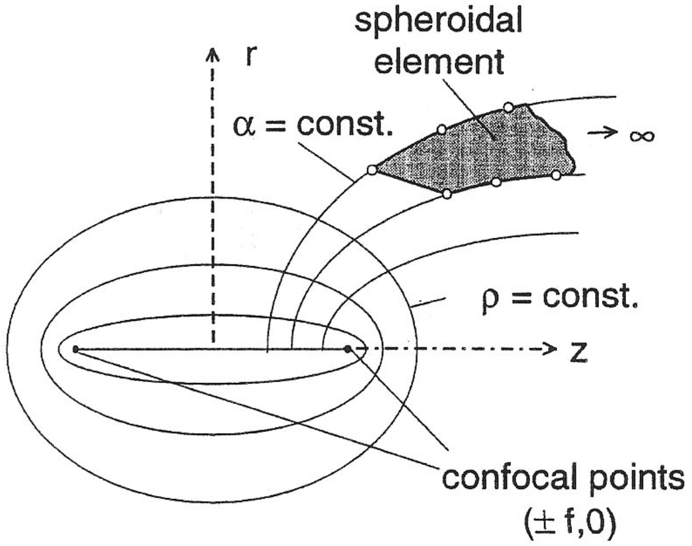


Milestones: Advanced Infinite Elements

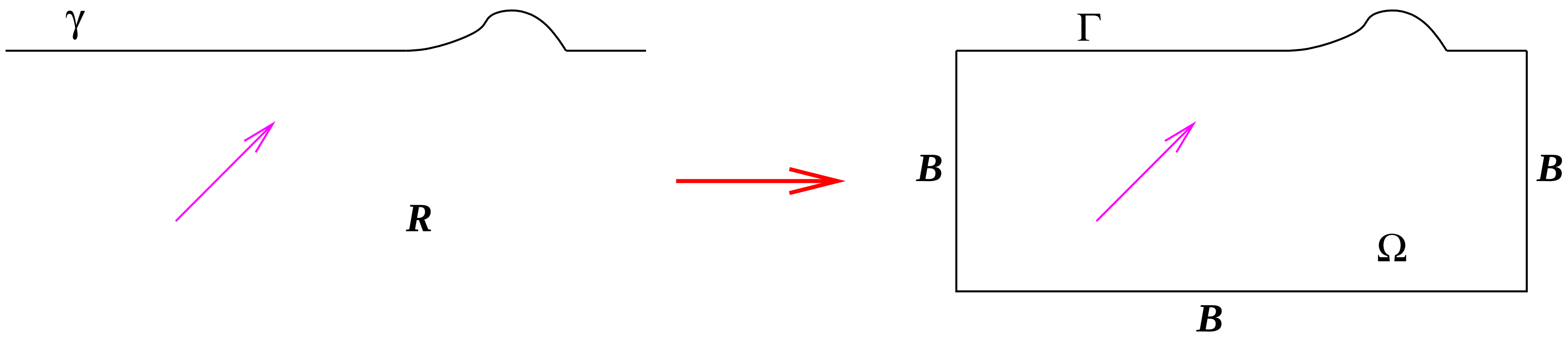
The 90's:

High Performance Infinite Elements for Waves

(Astley, Burnett)



Milestones: High-Order ABCs



Since mid 90's:

High-Order Local ABCs

- Collino [1993]: based on rational approximation.
- Grote & Keller [1995]: based on spherical harmonics.
- Hagstrom & Hariharan [1998]: based on the B-T ABC.
- Guddati & Tassoulas [2000]: based on rational approx.
- Givoli & Patlashenko [2001]: based on localized DtN.
- Givoli & Neta [2002]: based on the Higdon ABC.
- Hagstrom & Warburton [2004]: stabilized version of Givoli & Neta.

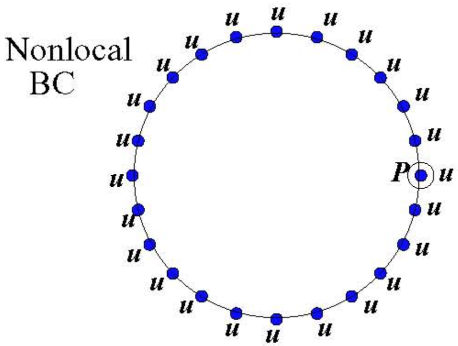
High-Order ABCs

Definition of High-Order ABC:

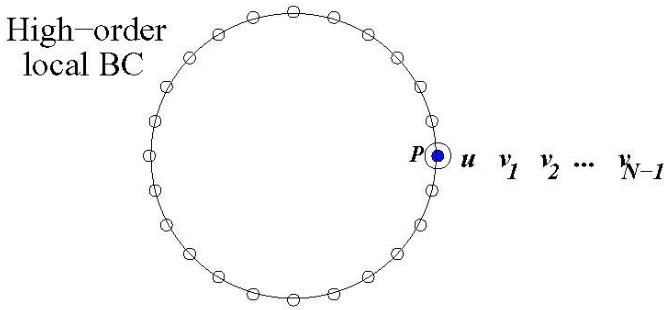
- Infinite sequence of ABCs with increasing accuracy
- Implementable for an arbitrarily high order!

High-Order ABCs Without High Derivatives:

•Nonlocal



•Auxiliary variables



Wave Dispersion

Waves with different frequencies (or wavelengths) travel with different speeds.

Dispersive wave equation:

$$\frac{\partial^2 u}{\partial t^2} = C_0^2 \nabla^2 u - f^2 u$$

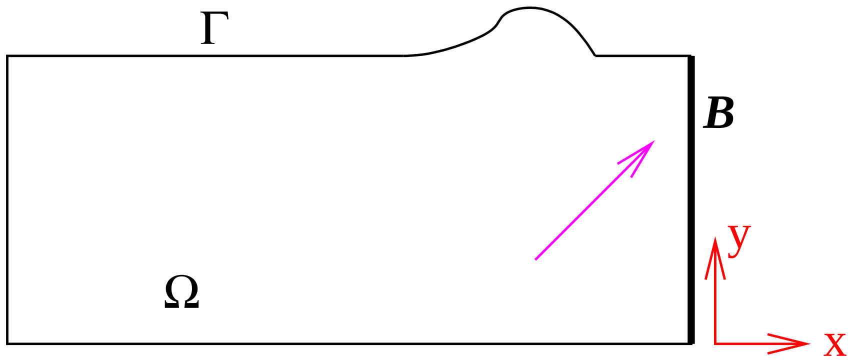
Ref. speed: $C_0 = \sqrt{gH_0}$

Phase speed: $C = C_0 \frac{1}{\sqrt{1-(f/\omega)^2}}$

Group speed: $C_g = C_0 \sqrt{1-(f/\omega)^2}$

$f = 0 \rightarrow$ Standard wave equation

The Higdon Absorbing Boundary Conditions
Higdon [SIAM J. Numer. Anal., 1990, 1994]



BC on *B*:

$$H_1: \quad \left(\frac{\partial}{\partial t} + C_1 \frac{\partial}{\partial x}\right) u = 0$$

$$H_2: \quad \left[\left(\frac{\partial}{\partial t} + C_2 \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + C_1 \frac{\partial}{\partial x}\right)\right] u = 0$$

$$\vdots$$

$$H_J: \quad \left[\prod_{j=1}^{J} \left(\frac{\partial}{\partial t} + C_j \frac{\partial}{\partial x}\right)\right] u = 0$$

$C_1, \ldots, C_J$: chosen parameters (phase velocities in the *x* direction)

The Higdon Absorbing Boundary Conditions (Contd.)

$$H_J : \quad \left[\prod_{j=1}^{J} \left(\frac{\partial}{\partial t} + C_j \frac{\partial}{\partial x} \right) \right] u = 0$$

Properties of the Higdon ABC

Good:

- Very general: any dimension, with or without dispersion or stratification.
- The H_J condition is exact for all plane waves moving with x-phase-speeds C_j, for $j = 1,\ldots,J$.
- Reflection coefficient for a wave with x-phase-speed C_x impinging on B:

$$R = \prod_{j=1}^{J} \left| \frac{C_j - C_x}{C_j + C_x} \right|$$

- Higdon's theorem: any rational-approx. ABC is either equivalent to H_J, or unstable, or non-optimal.

"Bad:"

- H_J developed in the literature up to $J = 3$ only.
- High-order x derivatives (only H_1 is compatible with low-order C^0 FEs).
- High-order t derivatives.
- No scheme provided in the literature for the *automatic* choice of the C_j.

The Higdon ABCs: High-Order Formulation

$$H_J: \quad \left[\prod_{j=1}^{J}\left(\frac{\partial}{\partial t}+C_j\frac{\partial}{\partial x}\right)\right]u=0$$

Introduce auxiliary variables ϕ_j:

$$\left(\frac{\partial}{\partial x}+\frac{1}{C_1}\frac{\partial}{\partial t}\right)u=\phi_1$$

$$\left(\frac{\partial}{\partial x}+\frac{1}{C_2}\frac{\partial}{\partial t}\right)\phi_1=\phi_2$$

$$\vdots$$

$$\left(\frac{\partial}{\partial x}+\frac{1}{C_J}\frac{\partial}{\partial t}\right)\phi_{J-1}=0$$

Define:

$$\phi_0 \equiv u, \phi_J \equiv 0 \quad \Rightarrow$$

$$\left(\frac{\partial}{\partial x}+\frac{1}{C_j}\frac{\partial}{\partial t}\right)\phi_{j-1}=\phi_j \qquad , \qquad j=1,\ldots,J$$

Need to get rid of the x (normal) derivatives!

The Higdon ABCs: High-Order Formulation (Contd.)

$$\left(\frac{\partial}{\partial x}+\frac{1}{C_j}\frac{\partial}{\partial t}\right)\phi_{j-1}=\phi_j \qquad , \qquad j=1,\dots,J$$

Each ϕ_j satisfies the dispersive wave equation:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{1}{C_0^2}\frac{\partial^2}{\partial t^2}-\frac{f^2}{C_0^2}\right)\phi_j=0$$

Use the identity:

$$\frac{\partial^2\phi_j}{\partial x^2}=\left(\frac{\partial}{\partial x}-\frac{1}{C_{j+1}}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x}+\frac{1}{C_{j+1}}\frac{\partial}{\partial t}\right)\phi_j+\frac{1}{C_{j+1}^2}\frac{\partial^2\phi_j}{\partial t^2}$$

BC on B:

$$\left(\frac{\partial}{\partial x}+\frac{1}{C_1}\frac{\partial}{\partial t}\right)u=\phi_1$$

$$\left(\frac{1}{C_j}+\frac{1}{C_{j+1}}\right)\frac{\partial\phi_j}{\partial t}=\phi_{j+1}$$

$$+\left(\frac{1}{C_j^2}-\frac{1}{C_0^2}\right)\frac{\partial^2\phi_{j-1}}{\partial t^2}+\frac{\partial^2\phi_{j-1}}{\partial y^2}-\frac{f^2}{C_0^2}\phi_{j-1}$$

Higdon's ABCs: The Parameters C_j

Three approaches for choosing the C_j:

(a) The user chooses the C_j a-priori.

(b) The C_j are chosen automatically by the code as a preprocess.

(c) The C_j are chosen and updated dynamically by the code.

Algorithm for approach (b):

- Estimate for the maximum *resolvable* wave number k_x:

$$(k_x)_{max} = \frac{\pi}{5\Delta x}$$

- Choose J-1 values of k from the interval $(0,(k_x)_{max})$, using the symm. minimax formula [Sommeijer et al.]

$$k_j = \left[\frac{(k_x)_{max}^2}{2} \left(1 + \cos\left(\frac{2j-1}{2(J-1)} \pi \right) \right) \right]^{\frac{1}{2}}, \qquad j = 1,\ldots,J-1$$

- Estimate the maximum *resolvable* wave number k_y:

$$(k_y)_{max} = \frac{\pi}{5\Delta y}$$

- For each k_j, calculate the corresponding frequency ω_j from the dispersion relation:

$$\omega_j = \sqrt{C_0^2 \left(k_j^2 + (k_y)_{max}^2 \right) + f^2}$$

- Calculate

$$C_j = \frac{\omega_j}{k_j} \qquad \text{for } j=1,\ldots,J-1.$$

- Add the value C_0 (the min. phase speed) to the J-1 values calculated above, thus generating J values.

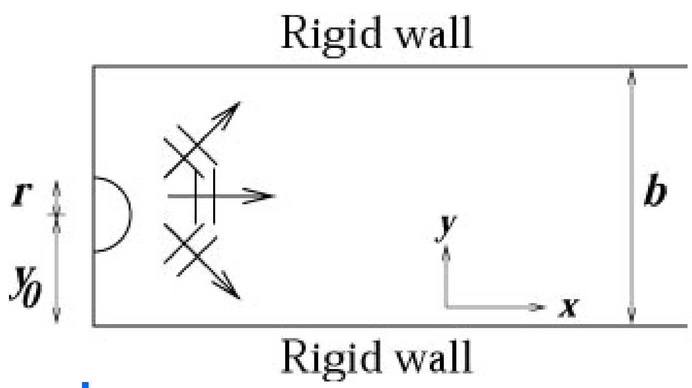
High-Order Higdon ABCs

Computational Aspects

- FE code: good for any J!
- Computational effort on B: linear with J
- Corners need special attention!
- The symmetric Hagstrom-Warburton formulation [2004] seems more stable than the Givoli-Neta formulation [2002]
- Convergence with J: theoretically at least like $1/J$, in practice exponential.
- Symmetric C^0 “mixed” FE formulation: stable with same shape function for all variables!

Dispersive Waves in a Channel: Example

Governing equations: Linearized SWEs



Wave source on the left boundary:

$$u_{left}(y,t) = \begin{cases} u_0 \cos \frac{\pi}{2r}(y - y_0) & \text{if} \quad |y - y_0| \le r \quad \& \quad t \le t_0 \\ 0 & \text{otherwise} \end{cases}$$

$y_0 =$ location, $r =$ width, $u_0 =$ amplitude, $t_0 =$ duration

Other parameters:

$$b = 5, \quad H_0 = 0.1, \quad g = 9.81 \quad \Rightarrow \quad C_0 = 0.99$$

Zero ICs

Time integration: Central diff. (explicit), $\Delta t = 0.025$

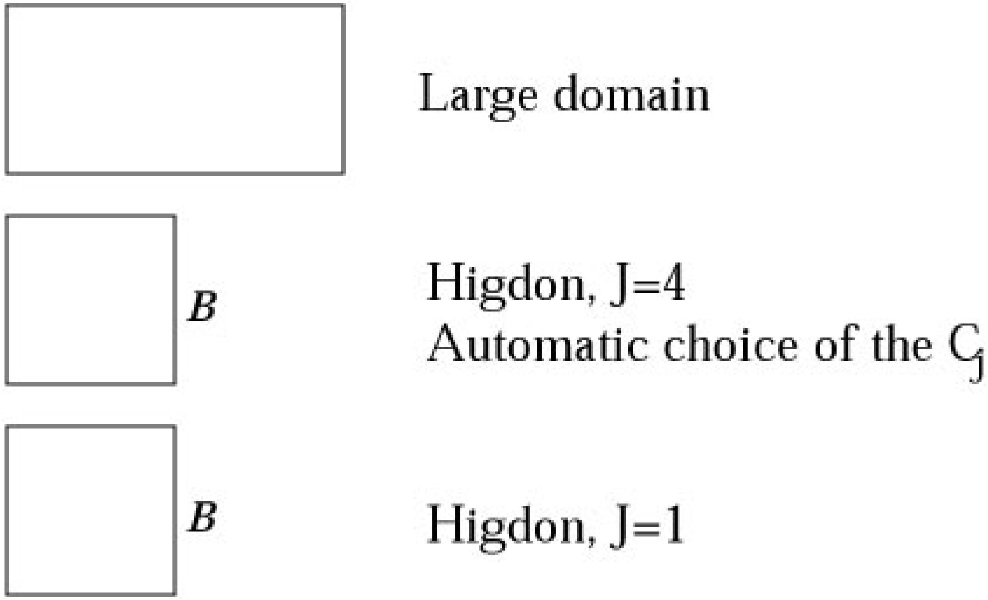
Dispersive Waves in a Channel: $f = 0.5$

Dispersion parameter: $f = 0.5$

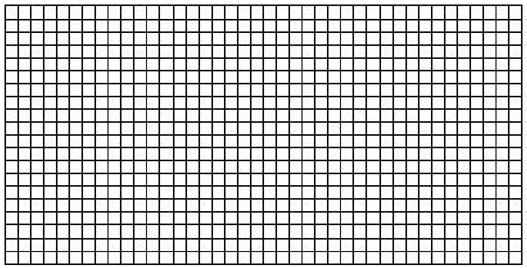
Source parameters:

$$y_0 = 2.5, \quad r = 1.5, \quad u_0 = 1, \quad t_0 = 0.5$$

Domains:



Dispersive Waves in a Channel: Meshes



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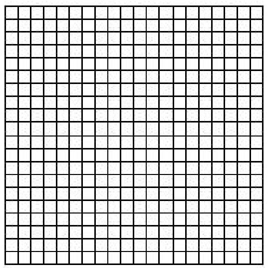
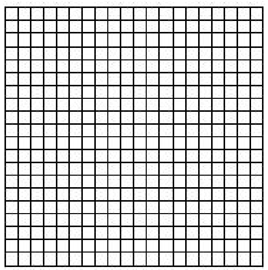
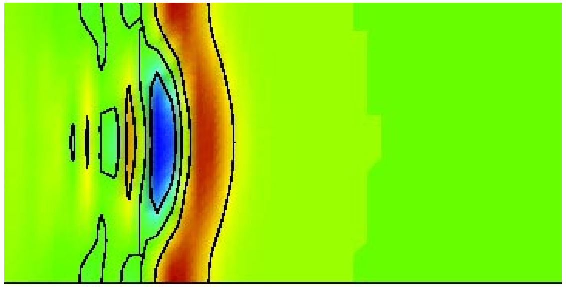


$J = 4$



$J = 1$

Results, $f = 0.5$, $t = 4$



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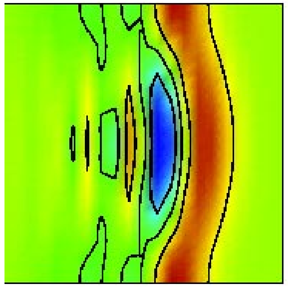
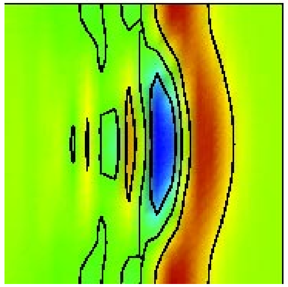
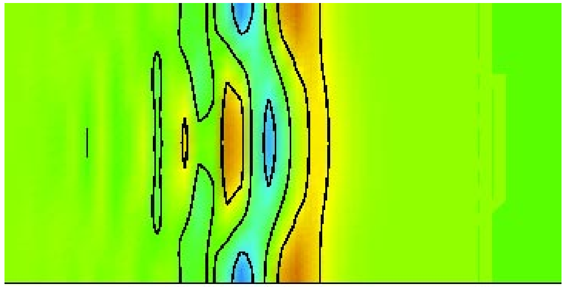


$J = 4$



$J = 1$

Results, $f = 0.5$, $t = 6$



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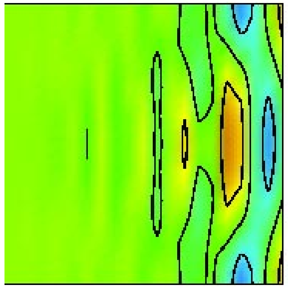
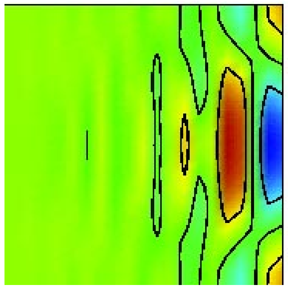
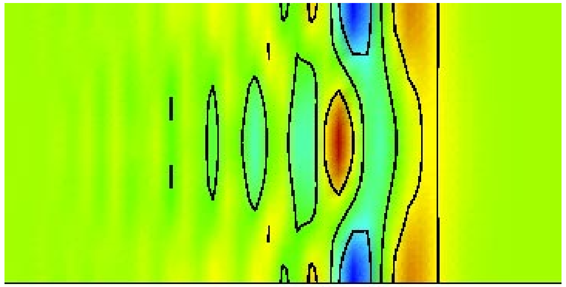


$J = 4$



$J = 1$

Results, $f = 0.5$, $t = 8$



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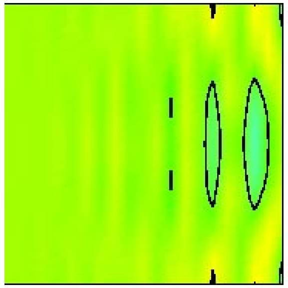
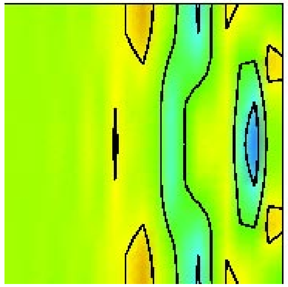
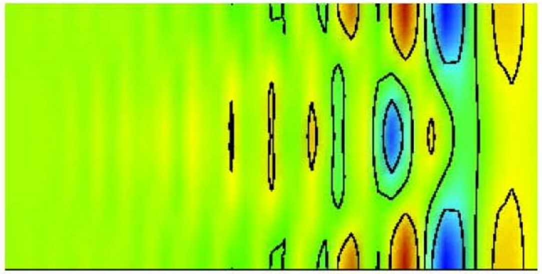


$J = 4$



$J = 1$

Results, $f = 0.5$, $t = 10$



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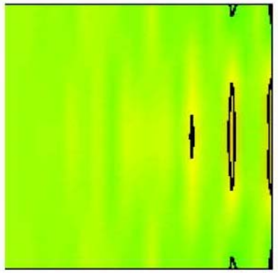
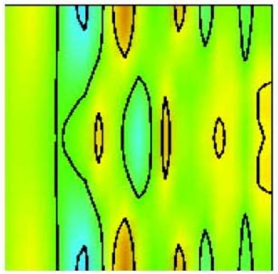


$J = 4$



$J = 1$

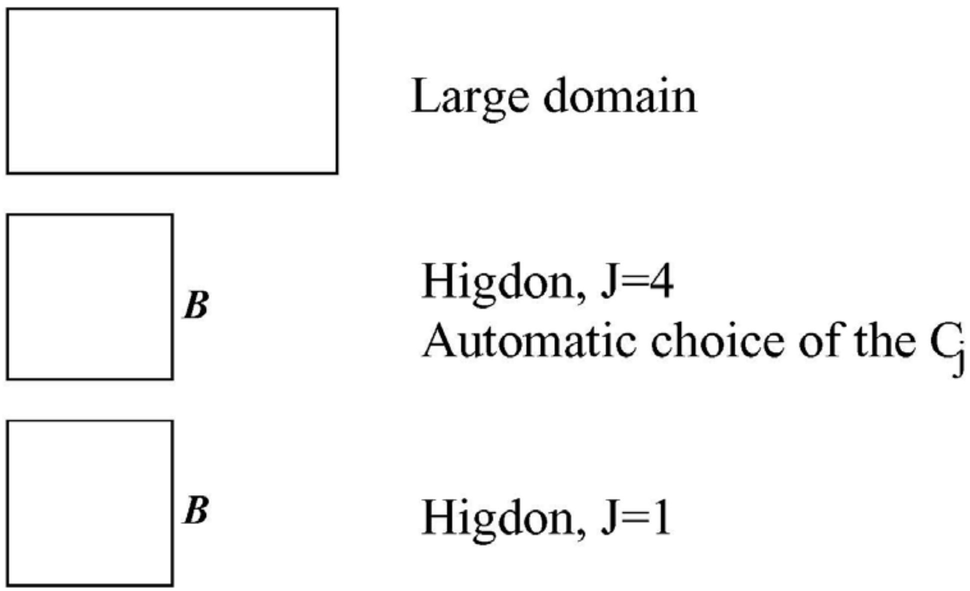
Dispersive Waves in a Channel: $f = 10$

Dispersion parameter: $f = 10$

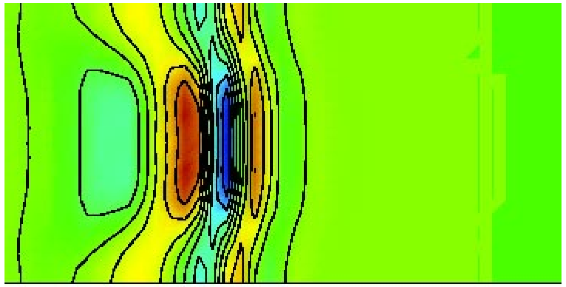
Source parameters:

$$y_0 = 2.5, \quad r = 1.5, \quad u_0 = 1, \quad t_0 = 0.5$$

Domains:



Results, $f = 10$, $t = 12.5$



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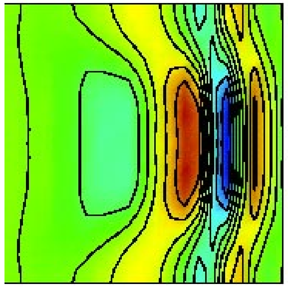

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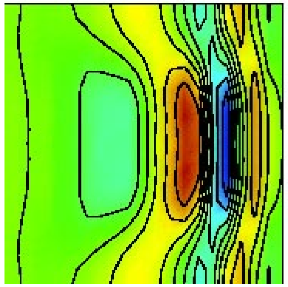
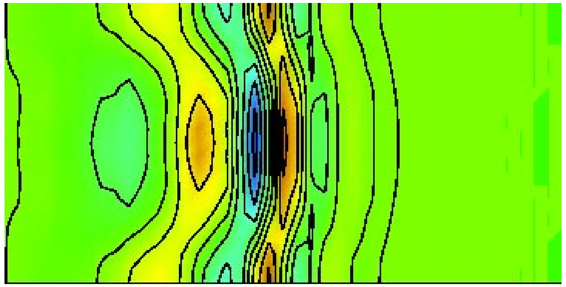


$J = 1$

Results, $f = 10$, $t = 15$



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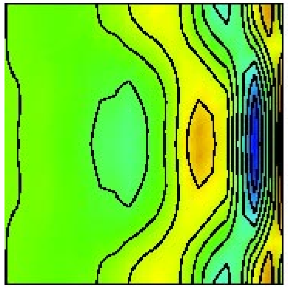
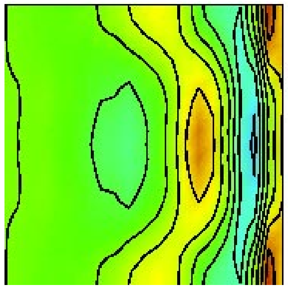
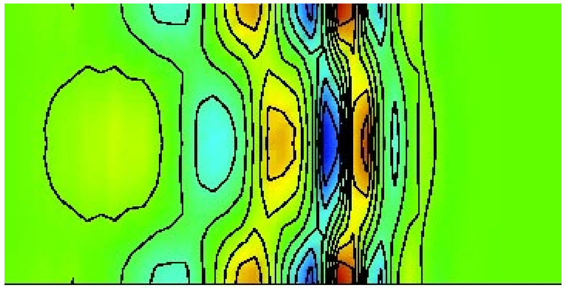


$J = 4$



$J = 1$

Results, $f = 10$, $t = 19$



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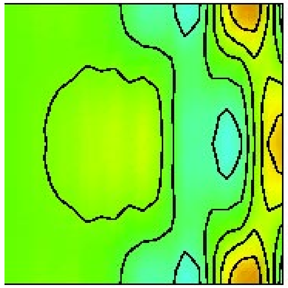
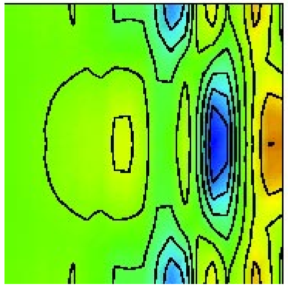
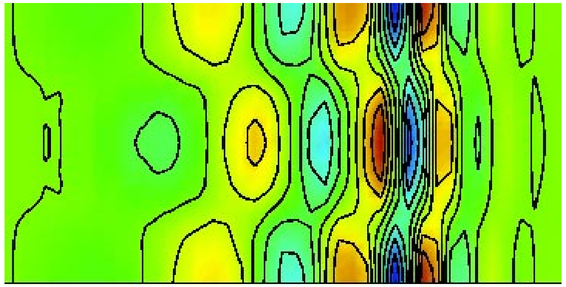


$J = 4$



$J = 1$

Results, $f = 10$, $t = 22.5$



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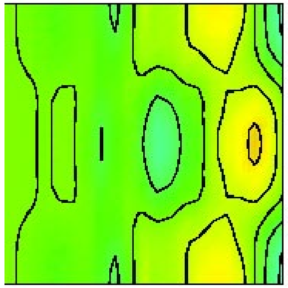
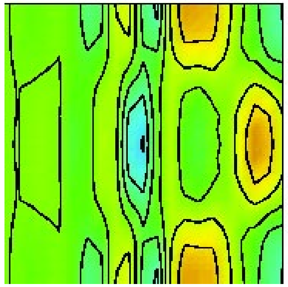


$J = 4$



$J = 1$

A Simplified Model in Meteorology [Van Joolen et al., IJNMF, 2004]

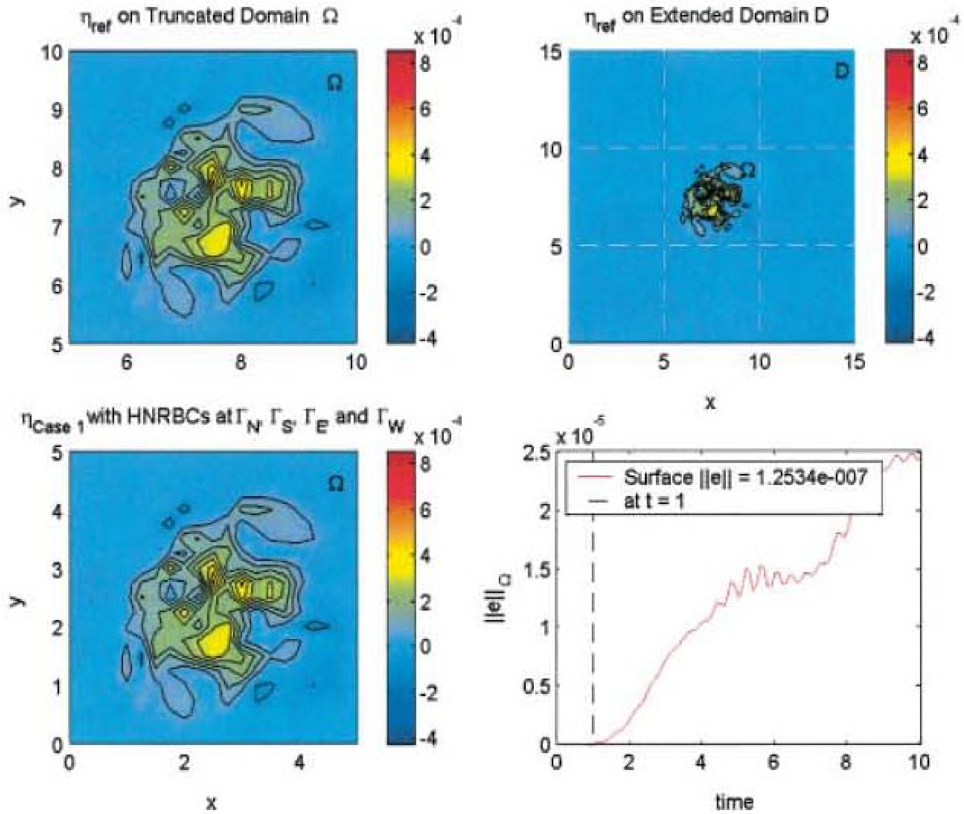


Plate 2. Single layer problem, the solution at $t = 1$ after the first event has been initiated.

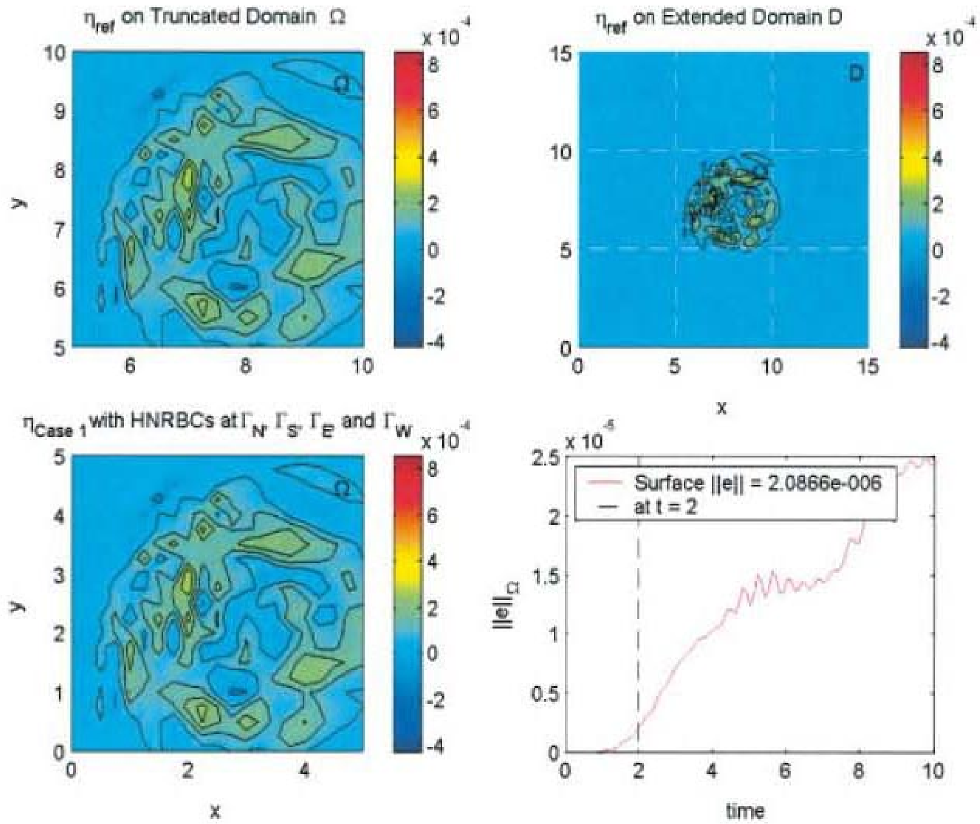


Plate 3. Single layer problem, the solution at $t = 2$ after the first event crosses Γ_S and Γ_E.

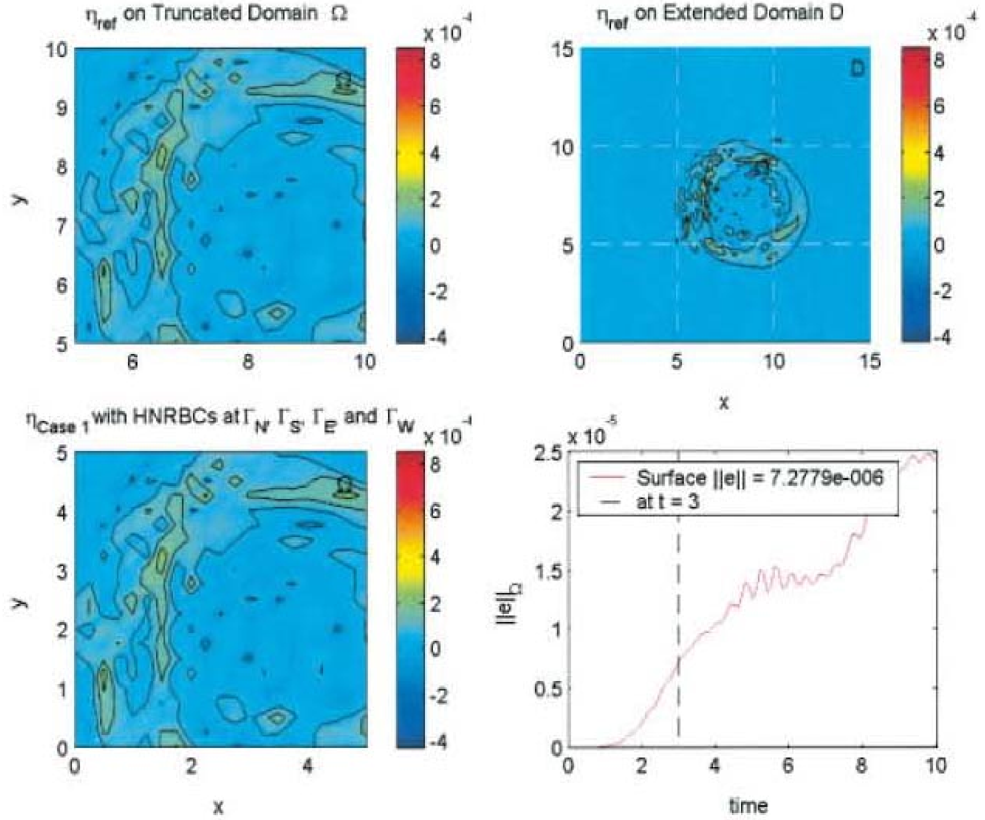


Plate 4. Single layer problem, the solution at $t = 3$ after the first event crosses Γ_N and Γ_W.

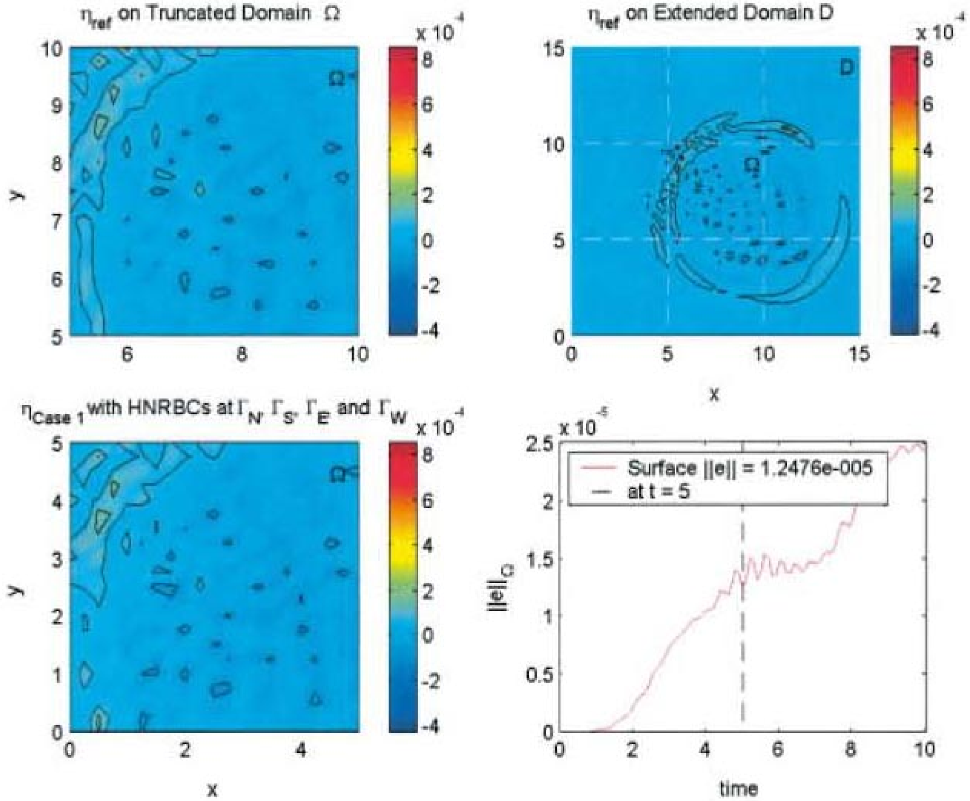


Plate 5. Single layer problem, the solution at $t = 5$ after the first event leaves Ω with visible spurious reflection at Γ_W.

CONCLUSION

- **Most Promising**
  - PML
  - High-Order ABCs

- **Current and Future Research**
  - More analysis (well-posedness, accuracy, numerical stability,…) needed for PML and high-order ABCs
  - Treatment of corner conditions
  - Treatment of heterogeneity in the exterior
  - Adaptivity of parameters and order
  - Incorporation of global data
  - Treatment of nonlinearity in the exterior