Waveform Sensitivity Kernels for General Aspherical Perturbations

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General formulation

Equation of motion for an earth model in the frequency domain:

$$\hat{H}\mathbf{u} - \rho\omega^2\mathbf{u} = \mathbf{f}\,. \tag{1}$$

For an SNREI earth we write

$$\hat{H}_0\mathbf{u}^0 - \rho_0\omega^2\mathbf{u}^0 = \mathbf{f}\,. \tag{2}$$

With a perturbation ansatz:

$$\begin{aligned} \hat{H} &= \hat{H}_0 + \hat{H}_1 & \rho &= \rho_0 + \rho_1 \\ \mathbf{u} &= \mathbf{u}^0 + \mathbf{v} & |\mathbf{v}| &\ll |\mathbf{u}| \end{aligned} \tag{3}$$

we obtain

$$\begin{aligned} (\hat{H}_0 - \rho_0\omega^2)\mathbf{v} &= -(\hat{H}_1 - \rho_1\omega^2)\mathbf{u} \\ (\hat{H}_0 - \rho_0\omega^2)\mathbf{v} &= -\hat{Z}\mathbf{u} \end{aligned} \tag{4}$$

On order to derive sensitivity kernels, we will later set the total field to $\mathbf{u}^0$, but this is not yet necessary. In this way the formula remains still exact:

$$(\hat{H}_0 - \rho_0\omega^2)\mathbf{v} = -\hat{Z}\mathbf{u}\,. \tag{5}$$

Free mode expansion and matrix elements

To make use of results derived by Woodhouse and Dahlen (1978) and Woodhouse (1980), we expand scattered and total displacement into eigenfunctions of the SNREI earth model:

$$\mathbf{u} = \sum_{k,m} u_{km}(\omega)\mathbf{s}_{km}(r,\theta,\phi)$$
$$\mathbf{v} = \sum_{k,m} v_{km}(\omega)\mathbf{s}_{km}(r,\theta,\phi)\,. \tag{6}$$

The eigenfunctions satisfy:

$$\hat{H}_0\mathbf{s}_{km} = \rho_0\omega_k^2\mathbf{s}_{km}\,, \tag{7}$$

where the eigenfunctions $\mathbf{s}_{km}$ can be of two different kinds, spheroidal and toroidal:

$$\mathbf{s}^S_{n\ell m} = \hat{\mathbf{D}}^S_{n\ell}(r)Y_{\ell m}(\theta,\phi) \qquad \mathbf{s}^T_{n\ell m} = \hat{\mathbf{D}}^T_{n\ell}(r)Y_{\ell m}(\theta,\phi)\,. \tag{8}$$

The $Y_{\ell m}(\theta,\phi)$ are spherical harmonics. The operator $\hat{\mathbf{D}}_{n\ell}(r)$ contains the radial eigenfunctions $U_{n\ell}(r)$, $V_{n\ell}(r)$ und $W_{n\ell}(r)$ of the earth model:

$$\hat{\mathbf{D}}^S_{n\ell}(r) = U_{n\ell}(r)\mathbf{e}_r + V_{n\ell}(r)\nabla_1 \qquad \hat{\mathbf{D}}^T_{n\ell}(r) = -W_{nl}(r)\mathbf{e}_r \times \nabla_1\,, \tag{9}$$

where $\nabla_1 = \mathbf{e}_\theta\frac{\partial}{\partial\theta} + \mathbf{e}_\phi\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}$ is the surface gradient.

Free mode expansion and matrix elements

The index k comprises the mode type, the radial overtone number n and the angular degree ℓ. The perturbation equation $(\hat{H}_0 - \rho_0\omega^2)\mathbf{v} = -\hat{Z}\mathbf{u}$ then reads

$$\begin{aligned} \sum_{km} v_{km}(\hat{H}_0 - \rho_0\omega^2)\mathbf{s}_{km} &= -\sum_{km} u_{km}\hat{Z}\mathbf{s}_{km} \\ \sum_{km} v_{km}(\rho_0\omega_k^2 - \rho_0\omega^2)\mathbf{s}_{km} &= -\sum_{km} u_{km}\hat{Z}\mathbf{s}_{km}\,. \end{aligned} \tag{10}$$

Since the eigenfunctions satsify an orthonormality relation:

$$\int_V \rho_0 \mathbf{s}^*_{k'm'} \cdot \mathbf{s}_{km}\, dV = \delta_{k'k}\delta_{m'm}\,, \tag{11}$$

multiplying the perturbation equation on both sides by $\mathbf{s}^*_{k'm'}$ and integrating over volume, yields

$$v_{k'm'}(\omega) = -\frac{1}{\omega_{k'}^2 - \omega^2} \sum_{km} u_{km}(\omega) \int_V (\mathbf{s}^*_{k'm'} \cdot \hat{Z}\mathbf{s}_{km})\, dV\,. \tag{12}$$

The integral expression on the right hand side is called a matrix element. Its explicit form has been derived by Woodhouse and Dahlen (1978) and Woodhouse (1980).

Generalized spherical harmonics

One of their basic result is that the matrix elements can be written as bilinear forms depending on $\mathbf{s}, \mathbf{s}^*, \nabla\mathbf{s}$ and $\nabla\mathbf{s}^*$:

$$\int_V (\mathbf{s}^*_{k'm'} \cdot \hat{Z}\mathbf{s}_{km})\, dV = Q_{k'km'm}(\mathbf{s}, \mathbf{s}^*, \nabla\mathbf{s}, \nabla\mathbf{s}^*)\,. \tag{13}$$

This fact can be used to find a general expression for the matrix elements. We use canonical coordinates with basis vectors

$$\mathbf{f}_{-1} = \frac{1}{\sqrt{2}}(\mathbf{e}_\vartheta - i\mathbf{e}_\varphi), \quad \mathbf{f}_0 = \mathbf{e}_r, \quad \mathbf{f}_{+1} = \frac{1}{\sqrt{2}}(-\mathbf{e}_\vartheta - i\mathbf{e}_\varphi)\,. \tag{14}$$

use of these basis vectors allows a generalization of the conventional spherical harmonics expansion to vector and tensor fields, for example the eigenfunctions can be expanded according to:

$$\mathbf{s}(r, \vartheta, \varphi) = \sum_{\alpha=-1}^{+1} U^\alpha_{\ell m}(r)\, Y^\alpha_{\ell m}(\vartheta, \varphi)\, \mathbf{f}_\alpha\,. \tag{15}$$

with an obvious generalization to higher order tensor fields.

Generalized spherical harmonics

The functions $Y_\ell^{\alpha,m}(\vartheta,\varphi)$ are called generalized spherical harmonics (GSH). They are useful to switch for example from a geographical to an epicentral reference frame with epicentral distance β and azimuth ξ:

$$Y_{\ell\sigma}(\beta,\xi) = \sum_m \overline{Y^\sigma_{\ell m}(\vartheta,\varphi)}\, Y_{\ell m}(\vartheta_R,\varphi_R)\,. \tag{16}$$

This is called the addition theorem of generalized spherical harmonics.
GSH can also be used to express the gradient of a vector field, for example:

$$\nabla \mathbf{s} = \sum_{\sigma=-1}^{+1}\sum_{\beta=-1}^{+1}\left(\delta_0^\beta\,\dot{U}^\sigma_{\ell m} + |\beta|\frac{1}{r}U^\sigma_{\ell m}\Omega_\ell^{\beta\sigma} - \frac{1}{r}|\beta|U^{\beta+\sigma}_{\ell m}\right)\, Y^{\beta+\sigma}_{\ell m}(\vartheta,\varphi)\,\mathbf{f}_\sigma\,\mathbf{f}_\beta\,, \tag{17}$$

with

$$\Omega_\ell^{\beta\sigma} = \sqrt{\frac{\ell(\ell+1)-\sigma(\sigma+\beta)}{2}}\,. \tag{18}$$

It becomes clear that the matrix element, which is a bilinear form in $\mathbf{s},\mathbf{s}^*,\nabla\mathbf{s}$ and $\nabla\mathbf{s}^*$, can be expanded into GSH according to the following general form:

$$\mathbf{s}^*_{k'm'}\cdot\hat{Z}\mathbf{s}_{km} = \gamma_{\ell'}\gamma_\ell\sum_{p=-2}^{+2}\sum_{q=-2}^{+2}\chi^{(pq)}_{k'k}(r,\vartheta,\varphi)\,Y^p_{\ell m}(\vartheta,\varphi)\,\overline{Y^q_{\ell' m'}(\vartheta,\varphi)}\,. \tag{19}$$

Important here is that the upper indices of the GSHs (p and q) take only values between -2 and 2! The expression $\gamma_\ell = \sqrt{\frac{2\ell+1}{4\pi}}$ has been extracted for convenience reasons.

Formulation as a scattering problem

We can use this representation of the matrix element to reformulate the equation

$$\begin{aligned} v_{k'm'}(\omega) &= -\frac{1}{\omega_{k'}^2 - \omega^2} \sum_{km} u_{km}(\omega) \int_V (\mathbf{s}_{k'm'}^* \cdot \hat{Z}\mathbf{s}_{km})\, dV \\ &= -\frac{\gamma_\ell'}{\omega_{k'}^2 - \omega^2} \sum_{km} u_{km}(\omega)\gamma_\ell \sum_{p,q=-2}^{+2} \int_V \chi_{k'k}^{(pq)}(r,\vartheta,\varphi) Y_{\ell m}^p(\vartheta,\varphi) \overline{Y_{\ell' m'}^q(\vartheta,\varphi)}\, dV \end{aligned} \tag{20}$$

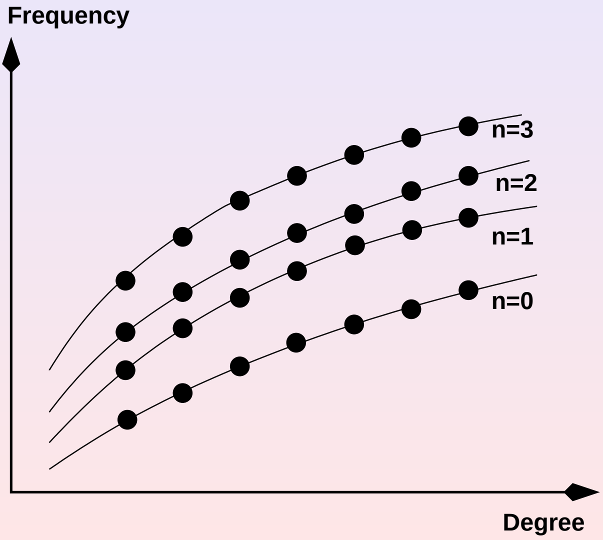
as a scattering problem.
First of all, we return to the scattered displacement at some receiver

$$\begin{aligned} \mathbf{v}(\vartheta_R,\varphi_R) &= \sum_{k'm'} v_{k'm'} \mathbf{s}_{k'm'} \\ &= -\sum_{k'm'} \frac{\gamma_\ell' \mathbf{s}_{k'm'}}{\omega_{k'}^2 - \omega^2} \sum_{km} u_{km}(\omega)\gamma_\ell \sum_{p,q=-2}^{+2} \int_V \chi_{k'k}^{(pq)}(r,\vartheta,\varphi) Y_{\ell m}^p(\vartheta,\varphi) \overline{Y_{\ell' m'}^q(\vartheta,\varphi)}\, dV \end{aligned} \tag{21}$$

To simplify things, we consider the scattered field of mode type t' and overtone branch n' excited by mode type t and overtone branch n:

$$\mathbf{v}_{n'n}^{t't}(\vartheta_R,\varphi_R) = -\sum_{\ell'm'} \frac{\gamma_\ell' \mathbf{s}_{n'\ell'm'}^t}{\omega_{n'\ell'}^2 - \omega^2} \sum_{\ell m} u_{km}(\omega)\gamma_\ell \sum_{p,q=-2}^{+2} \int_V \chi_{k'k}^{(pq)}(r,\vartheta,\varphi) Y_{\ell m}^p(\vartheta,\varphi) \overline{Y_{\ell' m'}^q(\vartheta,\varphi)}\, dV\,.$$

Formulation as a scattering problem



Formulation as a scattering problem

Now we insert the definition ($a =$ earth radius):

$$\mathbf{s}^{t}_{n\ell m}(a,\vartheta_R,\varphi_R) = \mathbf{D}^{t}_{n\ell}(a)\, Y_{\ell m}(\vartheta_R,\varphi_R)\,, \tag{22}$$

and obtain

$$\mathbf{v}^{t'\,t}_{n'\,n}(\vartheta_R,\varphi_R) = -\sum_{\ell' m'} \frac{\gamma'_{\ell}\mathbf{D}^{t}_{n'\ell'}(a)\, Y_{\ell' m'}(\vartheta_R,\varphi_R)}{\omega^2_{n'\ell'} - \omega^2} \times \tag{23}$$

$$\sum_{\ell m} u_{km}(\omega)\gamma_\ell \sum_{p,q=-2}^{+2} \int_V \chi^{(pq)}_{k'k}(r,\vartheta,\varphi) Y^{p}_{\ell m}(\vartheta,\varphi)\overline{Y^{q}_{\ell' m'}(\vartheta,\varphi)}\, dV\,.$$

We can now apply the addition theorem for GSH to introduce scatterer centered coordinates β and ξ giving the distance and azimuth of the receiver with respect to the scatterer:

$$Y_{\ell' q}(\beta,\xi) = \sum_{m'} \overline{Y^{q}_{\ell' m'}(\vartheta,\varphi)} Y_{\ell' m'}(\vartheta_R,\varphi_R)\,. \tag{24}$$

and rewrite the above equation as

$$\mathbf{v}^{t'\,t}_{n'\,n}(\vartheta_R,\varphi_R) = -\sum_{\ell'} \frac{\gamma'_{\ell}\mathbf{D}^{t}_{n'\ell'}(a)}{\omega^2_{n'\ell'} - \omega^2} \times \tag{25}$$

$$\sum_{p,q=-2}^{+2} \int_V Y_{\ell' q}(\beta,\xi) \sum_{\ell m} \chi^{(pq)}_{k'k}(r,\vartheta,\varphi)\, \gamma_\ell u_{km}(\omega)\, Y^{p}_{\ell m}(\vartheta,\varphi)\, dV\,.$$

Formulation as a scattering problem

This formula is easily interpreted: the total field at the scatterer excites a scattered signal whose strength is controlled by the interaction coefficients $\chi_{k'k}$. The signal is propagated to the receiver by the spherical harmonic $Y_{\ell' q}(\beta, \xi)$. We can still modify the formula bit. There exist generating operators G_ℓ^p with the following property:

$$\gamma_\ell Y_{\ell m}^p = \frac{\gamma_\ell}{\Theta_\ell^p} \hat{G}_\ell^p Y_{\ell m}^0 = \frac{1}{\Theta_\ell^p} \hat{G}_\ell^p Y_{\ell m} \,, \tag{26}$$

where

$$\Theta_\ell^p = \left[\frac{(\ell + |p|)!}{(\ell - |p|)!} \right]^{\frac{1}{2}} . \tag{27}$$

Using these operators we can write

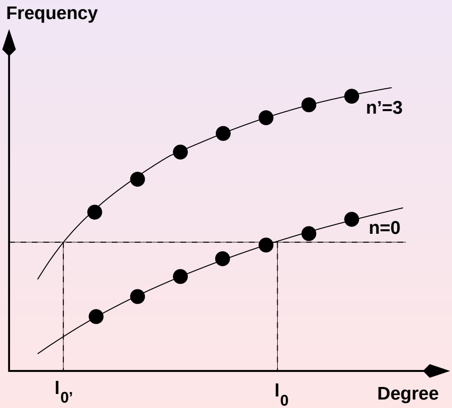
$$\mathbf{v}_{n'n}^{t't}(\vartheta_R, \varphi_R) = -\sum_{\ell'} \frac{\gamma_\ell' \mathbf{D}_{n'\ell'}^t(a)}{\omega_{n'\ell'}^2 - \omega^2} \times \tag{28}$$

$$\sum_{p,q=-2}^{+2} \int_V Y_{\ell' q}(\beta, \xi) \sum_\ell \chi_{k'k}^{(pq)}(r, \vartheta, \varphi) \frac{1}{\Theta_\ell^p} \hat{G}_\ell^p \sum_m u_{km}(\omega) Y_{\ell m}(\vartheta, \varphi) \, dV \,.$$

Applying the Poisson sum formula

Now, we are prepared to make the transition from modes to waves by applying the Poisson sum formula:

$$\sum_{\ell=0}^{\infty} g_\ell = \sum_{s=-\infty}^{\infty} \int_0^{\infty} g(\nu) e^{2\pi i s\nu} d\nu \,. \quad (29)$$



Applying the Poisson sum formula

Application of this formula to the sum over ℓ' and evaluation of the integral using the theorem of residues leads to the following expression:

$$\mathbf{v}_{n'n}^{t't}(\vartheta_R,\varphi_R) = -\frac{i}{4c'U'}\mathbf{D}_{n'\ell_0'}^{t}(a)\int_V dV\left(\frac{\beta}{\sin\beta}\right)^{1/2}\sum_{q=-2}^{+2}\frac{1}{\Theta_{\ell_0'}^{q}}H_{|q|}^{(2)}(z_0'\beta)[-s(q)\nu_0']^{|q|}e^{iq\xi}\times \tag{30}$$

$$\sum_{p=-2}^{2}\sum_{\ell}\chi_{k_0'k}^{(pq)}(r,\vartheta,\varphi)\frac{1}{\Theta_\ell^p}\hat{G}_\ell^p\sum_m u_{km}(\omega)Y_{\ell m}(\vartheta,\varphi)\,.$$

To apply the same procedure to the sum over ℓ, for simplicity we now set $\mathbf{u}=\mathbf{u_0}$. For an impulsive point source:

$$\sum_m u_{km}^0(\omega)Y_{\ell m}(\vartheta,\varphi) = \gamma_\ell\sum_{s=0}^{2}(-1)^s\frac{P_{\ell s}(\cos\vartheta)}{\omega_\ell^2-\omega^2}\mathrm{Re}\left[\left(\sqrt{2}e^{i\varphi}\right)^s q_{ks}\right]\,. \tag{31}$$

Applying the Poisson sum formula

Poisson formula and theorem of residues then yields

$$\mathbf{v}_{n'n}^{t't}(\vartheta_R,\varphi_R) = -\frac{i}{4c'U'}\mathbf{D}_{n'\ell_0'}^{t}(a)\int_V dV\left(\frac{\beta}{\sin\beta}\right)^{1/2}\sum_{q=-2}^{+2}\frac{1}{\Theta_{\ell_0'}^{q}}H_{|q|}^{(2)}(z_0'\beta)[-s(q)\nu_0']^{|q|}e^{iq\xi}\times \quad (32)$$

$$\sum_{p=-2}^{2}\chi_{k_0'k_0}^{(pq)}(r,\vartheta,\varphi)\frac{1}{\Theta_{\ell_0}^{p}}\hat{G}_{\ell_0}^{p}\times \quad (33)$$

$$\frac{-i}{4cU}\left(\frac{\vartheta}{\sin\vartheta}\right)^{1/2}\sum_{s=0}^{2}(-1)^s\nu_0^s H_s^{(2)}(z_0\vartheta)\mathrm{Re}\left[\left(\sqrt{2}e^{i\varphi}\right)^s q_{k_0 s}\right].$$

This is the general formula for calculating Born scattering seismograms for general aspherical perturbations. It can also be used to compute sensitivity kernels by restricting the support of the interaction coefficients $\chi_{k_0'k_0}^{(pq)}(r,\vartheta,\varphi)$ to a small volume and by computing the scattered field as a function of the location of the volume. It remains to evaluate the interaction coefficients for the desired kind of perturbations.

Interaction terms for general anisotropic perturbations

For anisotropic perturbations the matrix element takes the following form

$$\int_V (\mathbf{s}^*_{k'm'} \cdot \hat{Z}\mathbf{s}_{km})\, dV = \int_V \nabla\mathbf{s}^*_{k'm'} : \delta\mathbf{C} : \nabla\mathbf{s}_{km}\, dV \,. \tag{34}$$

Using a GSH representation of the eigenfunctions,

$$\mathbf{s}_{km} = \sum_{\alpha=-1}^{+1} U^{\alpha}_{km}(r) Y^{\alpha}_{km}\, \mathbf{f}_{\alpha} \,. \tag{35}$$

the integrand can be written

$$\nabla\mathbf{s}^*_{k'm'} : \delta\mathbf{C} : \nabla\mathbf{s}_{km} = \sum_{\alpha\beta}\sum_{\rho\tau} \overline{U^{\alpha|\beta}_{k'm'}}\, (-1)^{\rho+\tau} \delta C^{\beta\alpha,-\rho,-\tau}\, U^{\tau|\rho}_{km} Y^{\rho+\tau}_{km} \overline{Y^{\beta+\alpha}_{k'm'}} \tag{36}$$

with

$$U^{\tau|\rho}_{km} = \left(\delta^{\rho}_{0}\, \dot{U}^{\tau}_{km} + |\rho| \frac{1}{r} U^{\tau}_{km} \Omega^{\rho\tau}_{\ell} - \frac{1}{r} |\rho| U^{\rho+\tau}_{km} \right) \,. \tag{37}$$

These equations allow an evaluation of the interaction terms from the radial eigenfunctions of a spherically symmetric earth model. Explicit expressions are skipped here but we note some symmetry properties.

Symmetry properties

Setting $p = \rho + \tau$ and $q = \alpha + \beta$ one can write the interaction coefficients in the form

$$\chi^{p,q}_{k'k} = \overline{U^{q-\beta|\beta}_{k'm'}}(-1)^p \delta\mathbf{C}^{\beta,q-\beta,-\rho,-p+\rho} U^{p-\rho|\rho}_{km}\,. \tag{38}$$

Due to the symmetry relations (not proven here)

$$U^{-\alpha|-\beta}_{km} = \overline{U^{\alpha|\beta}_{km}} \tag{39}$$

and

$$\delta\mathbf{C}^{\alpha\beta\gamma\delta} = (-1)^{\alpha+\beta+\gamma+\delta}\overline{\delta\mathbf{C}^{-\alpha,-\beta,-\gamma,-\delta}}\,, \tag{40}$$

it can be shown that

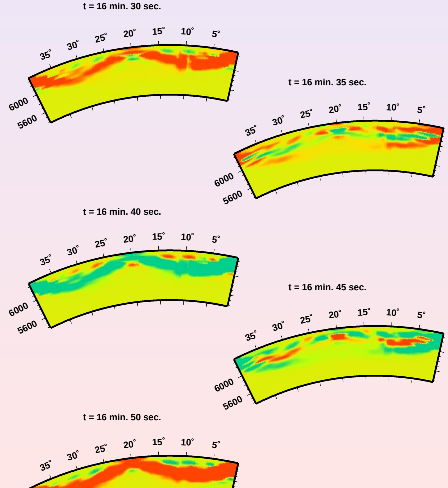
$$\overline{\chi^{-p,-q}_{k'k}} = (-1)^{p-q}\chi^{pq}_{k'k}\,. \tag{41}$$

Thus, from the 25 interaction coefficients we only need to compute 13. Since the sum of the indices of the elastic tensor is $q - p$ we order the interaction coefficients according to this quantity. An evaluation shows that there are

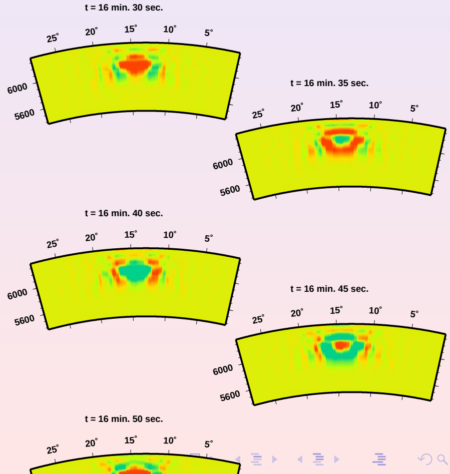
- 3 coefficients with $q - p = 0$. They contain five elastic constants which represent transversely isotropic perturbations
- 4 coefficients with $q - p = 1$. They contain 6 elastic constants.
- 3 coefficients with $q - p = 2$. They contain 6 elastic constants.
- 2 coefficients with $q - p = 3$. They contain 2 elastic constants.
- 1 coefficient with $q - p = 4$. It contains 2 elastic constants.

Spheroidal-spheroidal coupling, transversely isotropic perturbation, symmetric radiation pattern

Great circle section:

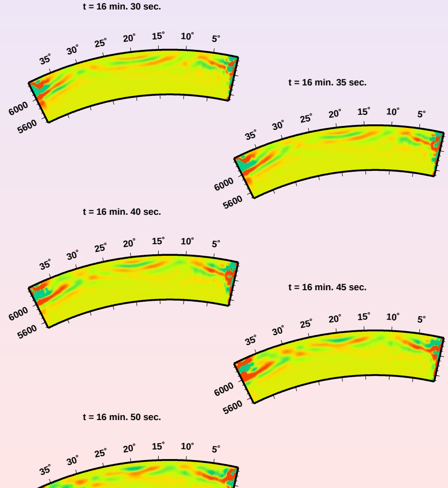


Cross section perpendicular to great circle:

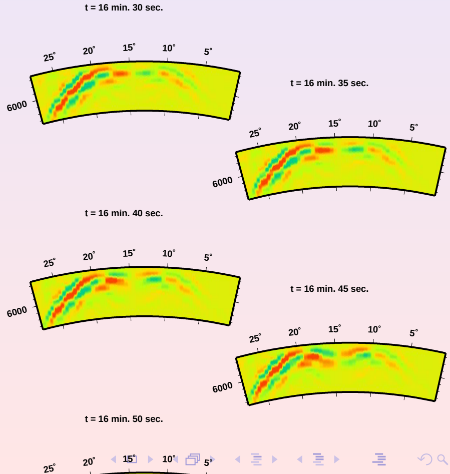


Toroidal-spheroidal coupling, transversely isotropic perturbation, non-symmetric radiation pattern

Great circle section:

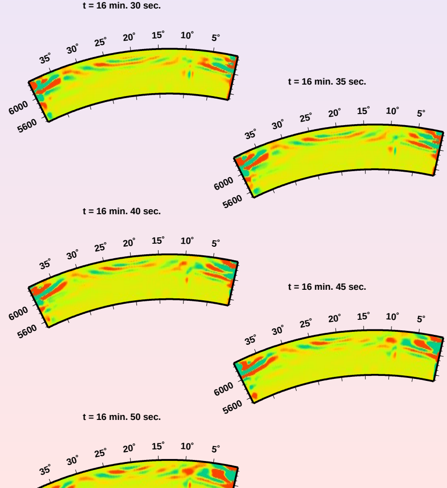


Cross section perpendicular to great circle:

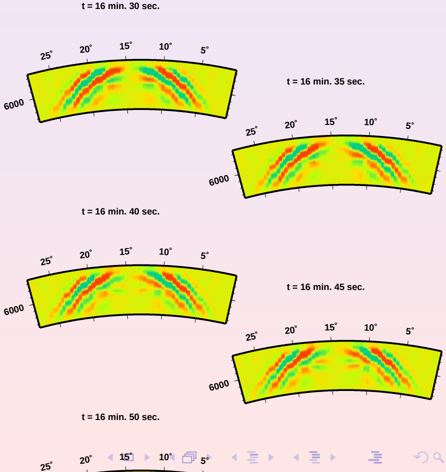


Spheroidal-toroidal coupling, transversely isotropic perturbation, symmetric radiation pattern

Great circle section:

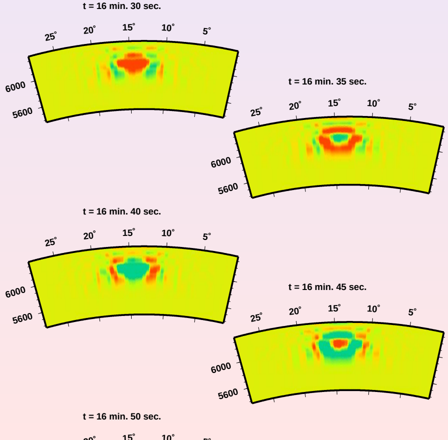


Cross section perpendicular to great circle:



Spheroidal-spheroidal coupling, purely anisotropic perturbation $\delta C^{1111}, \delta C^{-1,-1,-1,-1}$, symmetric radiation pattern

Great circle section:



Cross section perpendicular to great circle: