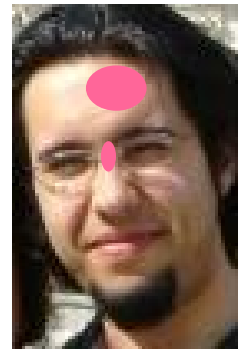
Seismic wave Propagation and Imaging in Complex media: a European network



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Early Stage Researcher

Host Institution: LMU Munich.
Place of Origin: Barcelona, Spain.
Appointment Time: October 2004.

Project: Simulation of Earthquakes on Unstructured Meshes, Viscoelasticity, Anisotropy, Poroelasticity, ADER-DG Method.

Task Groups: TG Numerical Methods, TG Digital Library.

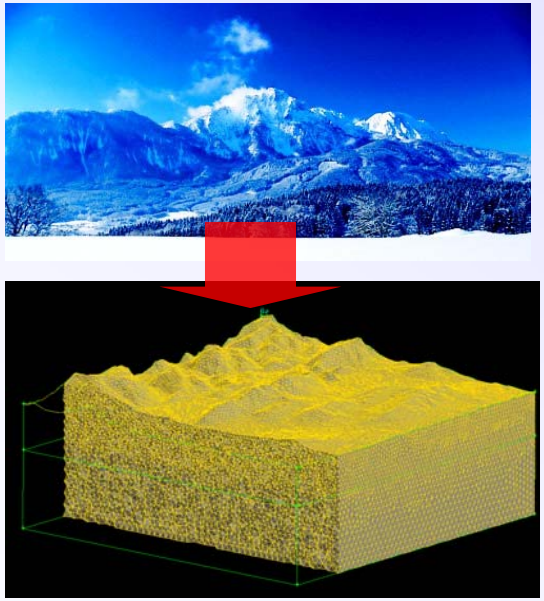
Cooperation: Trento University.

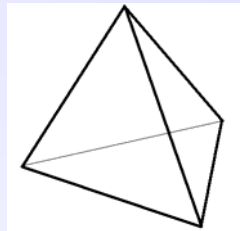
Project Scope: *Unstructured Meshes in Seismology*

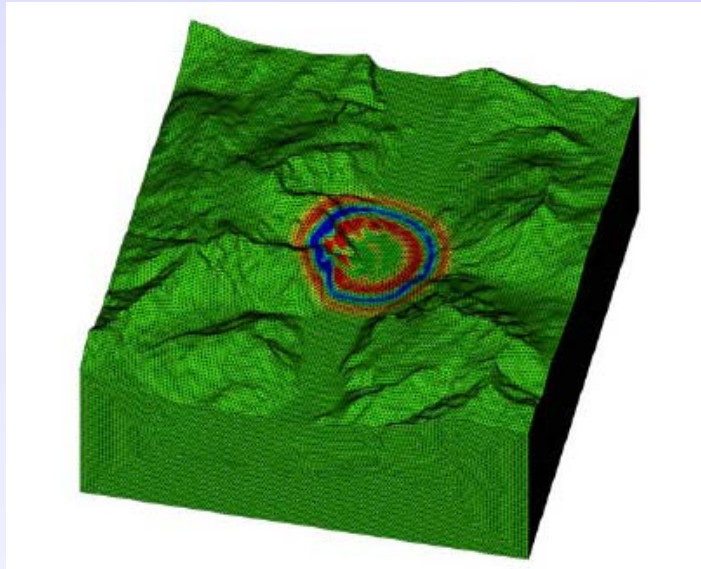
We are developing a method to simulate seismic wavefields on tetrahedral unstructured meshes.

The ADER-DG (Arbitrary high order DERivatives-Discontinuous Galerkin) method is born as a combination of the most convenient features of higher-order finite elements (FE) and finite volumes (FV).

In the last two years and a half it has been adapted to the problem of computational seismology.



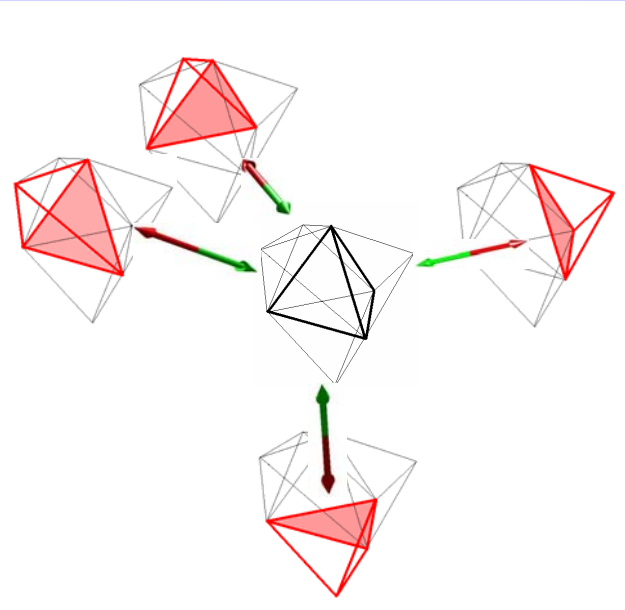




The ADER-DG Method (Käser & Dumbser)

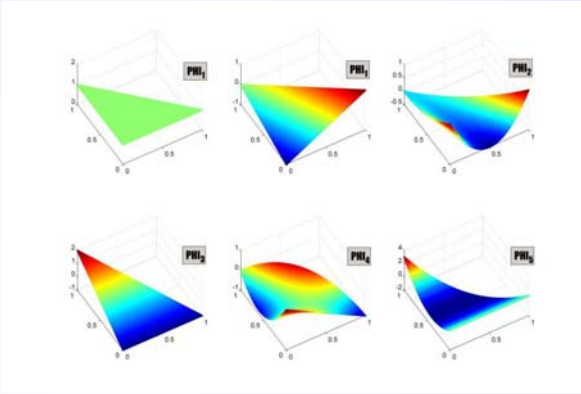
Some aspects of the method:

- Uses tetrahedra as computational cells.
- Polynomial basis functions inside each element.
- Communication between cells through "fluxes".
- High-order single-step time integration (ADER).
- Possibility of using local timestepping (LTS).



Currently implemented functionalities:

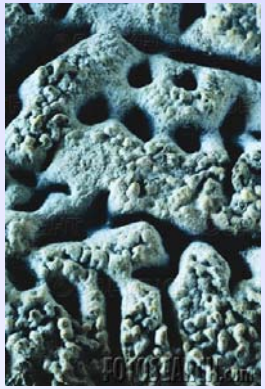
- Viscoelasticity.
- Kinematic finite source models.
- Anisotropy.
- Fluid/solid interfaces.



Time integration is fully explicit.
Parallelization has been tested up to 1024 CPUs.
Upcoming hexahedral version.

…check "SPICE Code Validation" web for accuracy tests!

Poroelasticity



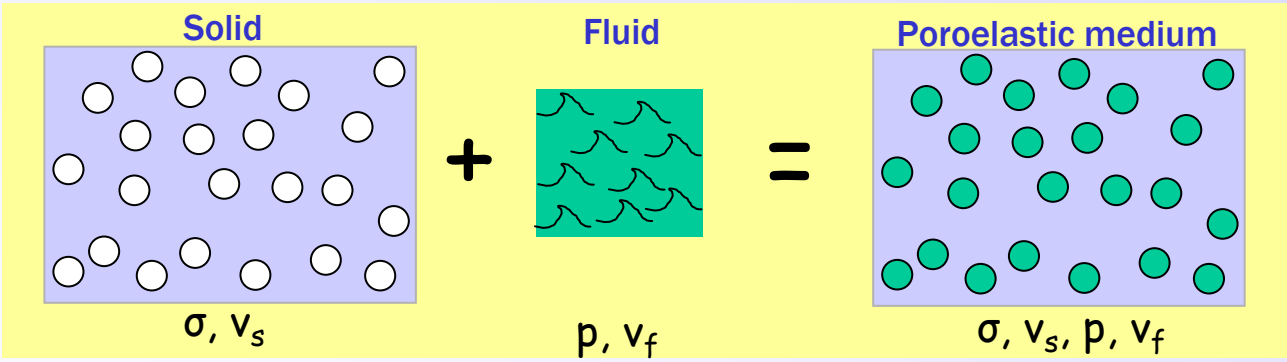
Poroelasticity is a theory aiming at describing the mechanical properties of porous rocks filled with some fluid.

The main development of the theory is acknowledged to M.A. Biot in the 40s, which applies to fluid-filled porous materials with interconnected pores. The pores are supposed to be much smaller than the wavelengths investigated (continuum mechanics).

The properties of a poroelastic material are a combination of the properties of:

- The mineral itself (solid or grain).
- The dry porous rock (matrix).
- The fluid filling the pores (fluid).

The particle motion is macroscopically described as that of the solid and of the fluid.



Parameters defining a porous material:

- **Solid**
  - K_S: bulk modulus
  - ρ_S: density
- **Matrix**
  - c_{ij}: Hooke's matrix components
  - ϕ: porosity
  - k: permeability
  - T: tortuosity
- **Fluid**
  - K_F: bulk modulus
  - ρ_F: density
  - η: viscosity

$$M = \frac{K_S}{(1 - K_M / K_S) - \phi(1 - K_S / K_F)}$$

$$\alpha = 1 - K_M / K_S$$

$$c_{ij}^u = c_{ij} + M\alpha^2$$

$$\rho = (1-\phi)\rho_S + \phi\rho_F$$

$$m = \mathrm{T}\rho_F / \phi$$

Similarly, a new constitutive relation relates stresses (and pore pressure p) to strains (and variation of fluid content ζ):

$$\sigma_{ij} = c_{ijkl}^u \varepsilon_{kl} - M\alpha\zeta$$

$$p = M\left(\zeta - \alpha \cdot tr(\varepsilon_{ij})\right)$$

$$\zeta \equiv -\nabla \cdot \left[\phi\left(u - u^f\right)\right]$$

Those have to be combined to a new set of dynamic equations: Darcy's laws and Biot-Euler equations:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial u_i}{\partial t} + \rho_F \frac{\partial u_i^{(f)}}{\partial t}$$

$$\frac{\partial p}{\partial x_i} = -\rho_F \frac{\partial u_i}{\partial t} - \Psi * \frac{\partial u_i^{(f)}}{\partial t}$$

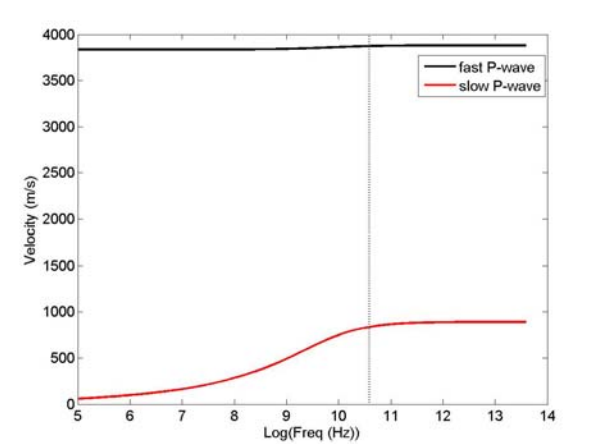
$$\Psi(t) \equiv m\delta(t) + \frac{\nu}{\kappa} H(t)$$

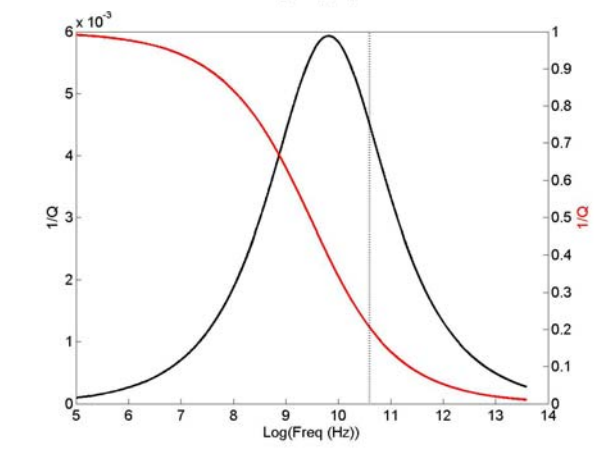
To obtain a wave propagation equation system for poroelastic materials.

A first result of the poroelastic theory is the existence of a new compressional wave („Biot's slow wave") with a much smaller propagation velocity.

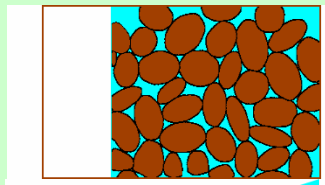
Viscosity of the liquid phase produces intrinsic energy losses. Also a clear distinction between high- and low-frequency regimes.

$$f_c = \frac{\nu\phi}{\mathrm{T}\kappa\rho_F}$$

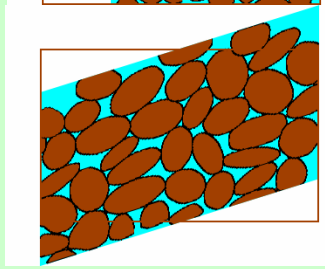




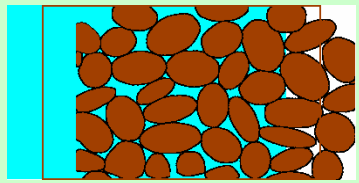
Fast P-wave



S-wave



Slow P-wave



Poroelasticity in ADER-DG Schemes

Introducing poroelasticity in ADER-DG schemes is basically two jobs in one: the high-frequency and the low frequency.

High-frequency:

All modes are propagatory, although appears a new wave type. Mesh sampling has to be adjusted to resolve its very small wavelengths. The essentials of the **ADER-DG** method remain unchanged.

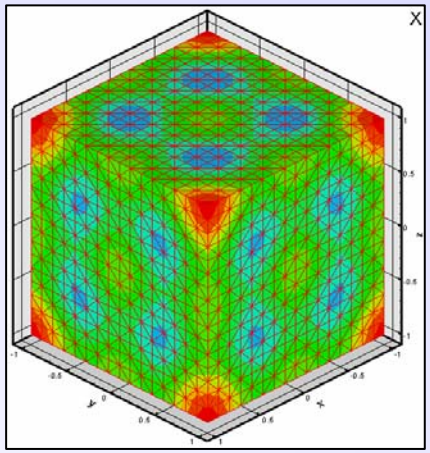
Low-frequency:

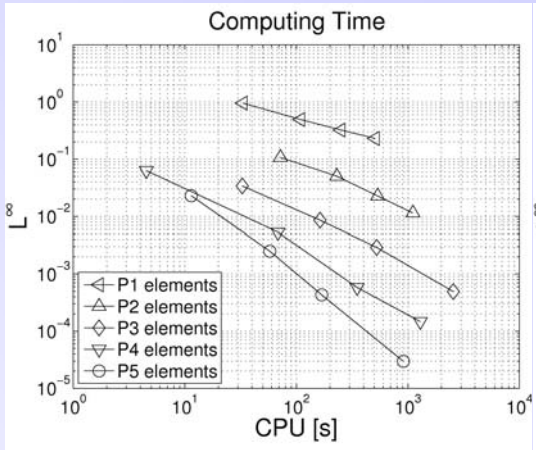
The equations become „stiff", meaning that two largely different timescales are involved. Most explicit solvers are unable to handle with such problems. As a consequence we try out two approaches:

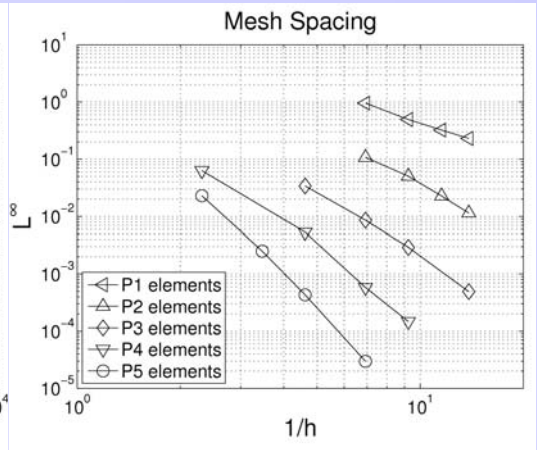
- **ADER-DG(FS):** Using the well known fraction-step method, based upon equation system splitting.
- **ADER-DG(ST):** Space-Time Discontinuous method, novel approach resulting in a locally implicit and globally explicit method.

High-Frequency Validation:

We first prove the convergence properties of the method by using a periodic problem of plane wave propagation combining simultaneously all 3 wave types in 3 perpendicular directions.

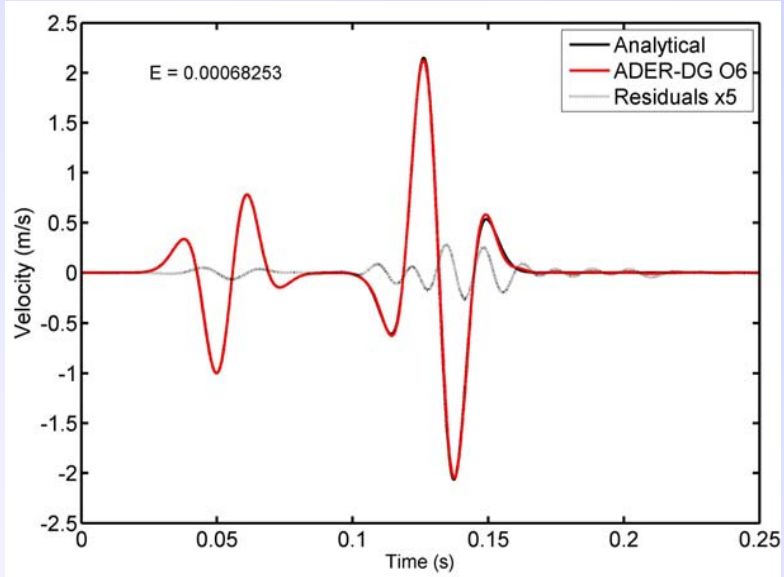






To asses misfits in recorded seimograms we compare to analytical solutions as the one from Dai et al. (1995) for a point explosion.

- Ricker wavelet 30Hz, homogeneous inviscid poroelastic material.
- Offset=100m.
- Cf = 6916 m/s
- Cs = 1092 m/s
- No S-waves are generated.
- ADER-DG *O*6 simulation on 455,625 tetrahedra
- 900m x 900m x 900m domain, Δx~20m.

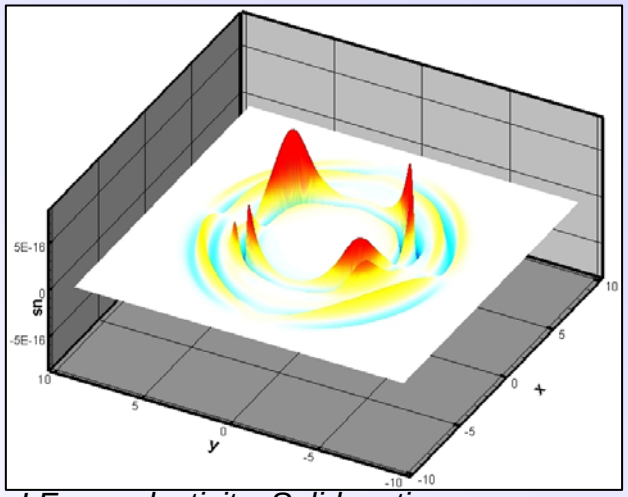


Low frequency: ADER-DG(FS)

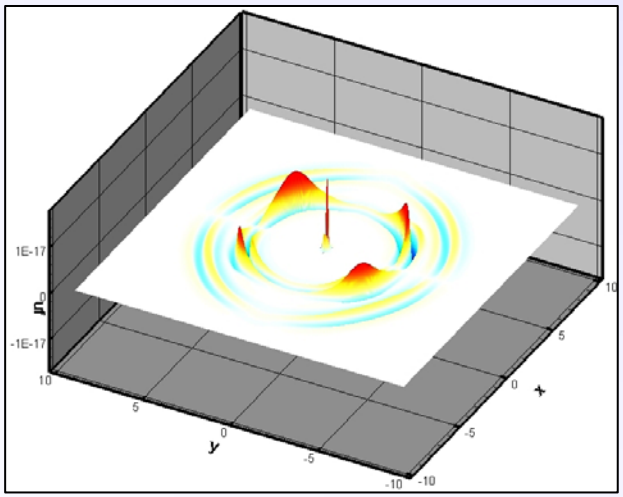
Equation system to be solved:

$$\frac{\partial Q_p}{\partial t} + A_{pq}\frac{\partial Q_q}{\partial x} + B_{pq}\frac{\partial Q_q}{\partial x} + C_{pq}\frac{\partial Q_q}{\partial x} = E_{pq}Q_q$$

stiff!!



LF poroelasticity: Solid motion



LF poroelasticity: Fluid motion

The „fraction-step method" involves the separation of the equation in two parts, which are solved independently in an alternating way.

$$\begin{cases} \dfrac{\partial Q_p}{\partial t} + A_{pq}\dfrac{\partial Q_q}{\partial x} + B_{pq}\dfrac{\partial Q_q}{\partial x} + C_{pq}\dfrac{\partial Q_q}{\partial x} = 0 \\ \dfrac{\partial Q_p}{\partial t} = E_{pq}Q_q \end{cases}$$

The *stiff* term can be solved analytically:

$$\overset{\rho}{v}{}^{*}_{f}(\Delta t) = \exp\left(\frac{-\beta_1 \upsilon}{\rho_1 \kappa}\Delta t\right)\overset{\rho}{v}_{f}(0)$$

$$\overset{\rho}{v}{}^{*}(\Delta t) = \frac{\beta_1 \rho_2}{\beta_2 \rho_1}\exp\left(\frac{-\beta_2 \upsilon}{\rho_2 \kappa}\Delta t - 1\right)\overset{\rho}{v}_{f}(0) + \overset{\rho}{v}{}^{*}_{f}(0)$$

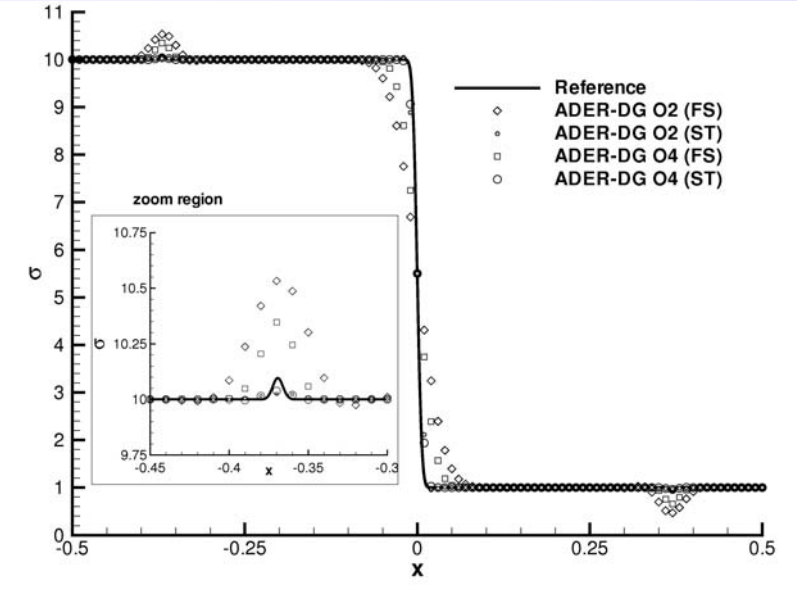
The partitioned result is 1st order accurate, but the ↳t is as large as that of the ADER-DG non-stiff explicit time-solver.

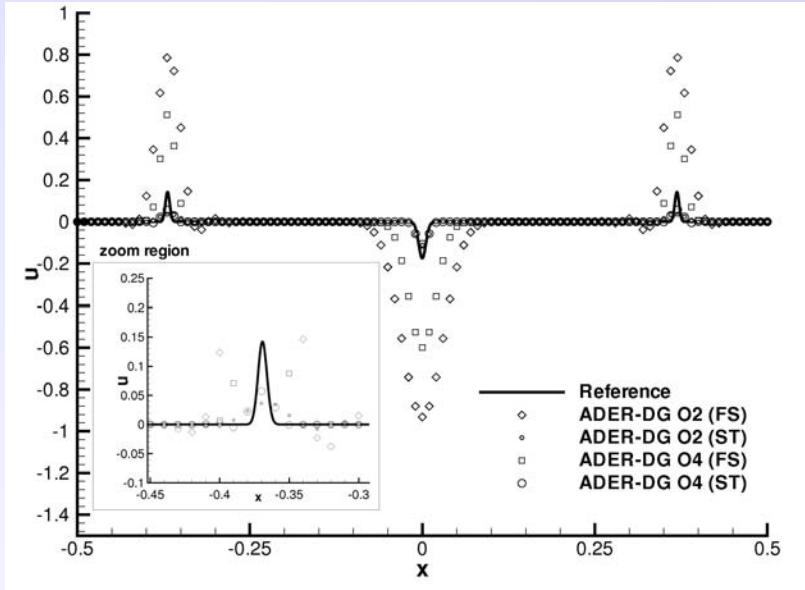
Low frequency: ADER-DG(ST)

An alterative is the use of a *local* Space-Time Discontinuous Galerkin method as introduced by Dumbser & Enaux (2006).

Conceptually means that we allow our solution to be *discontinuous* between two time-steps, and the discontinuity is then solved through numerical fluxes. This involves a system matrix inversion for each local element, as opposed to inverting a global system matrix. Therefore the scheme can be considered locally implicit although globally it is an explicit method.

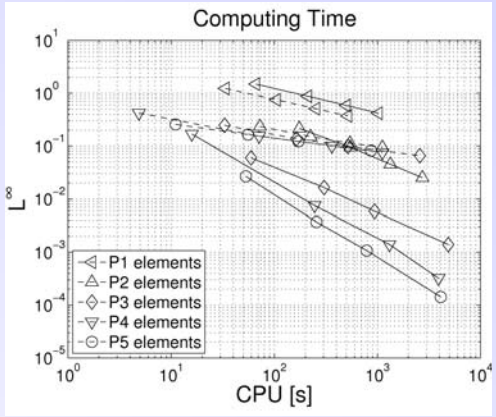
1D comparison for a poroelastic material of very filled with very high viscosity fluid:

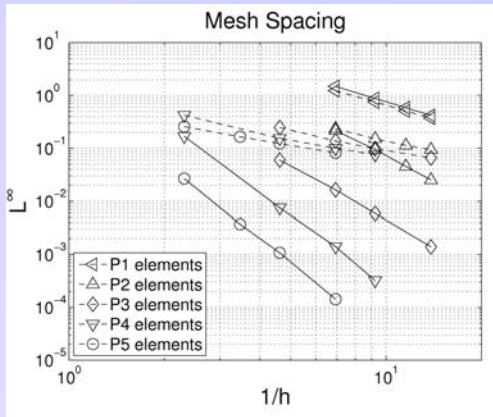


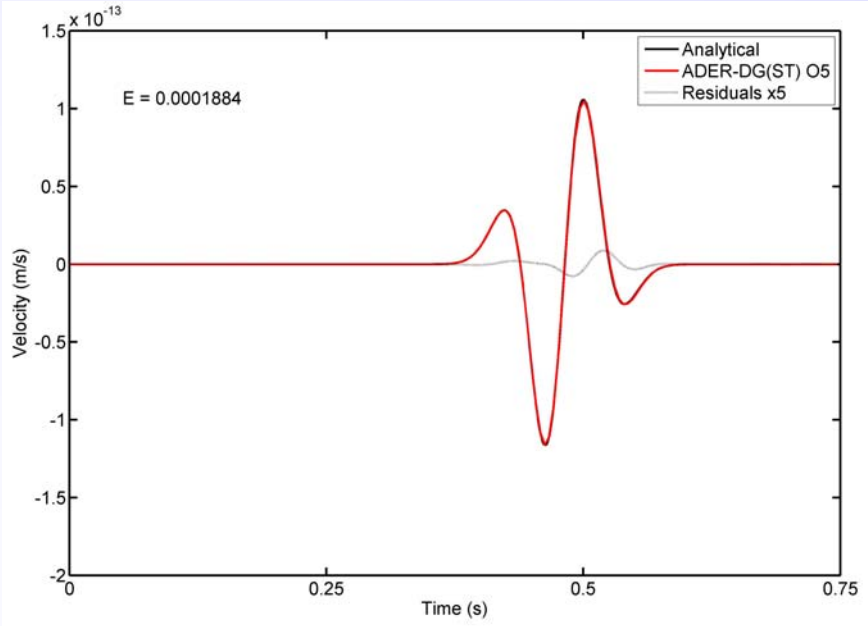


Low frequency Validation:

Again we first compute convergence tests, now showing the clear improvement in convergence of ADER-DG(ST) (solid line) over the first-order ADER-DG(FS) (dashed line):







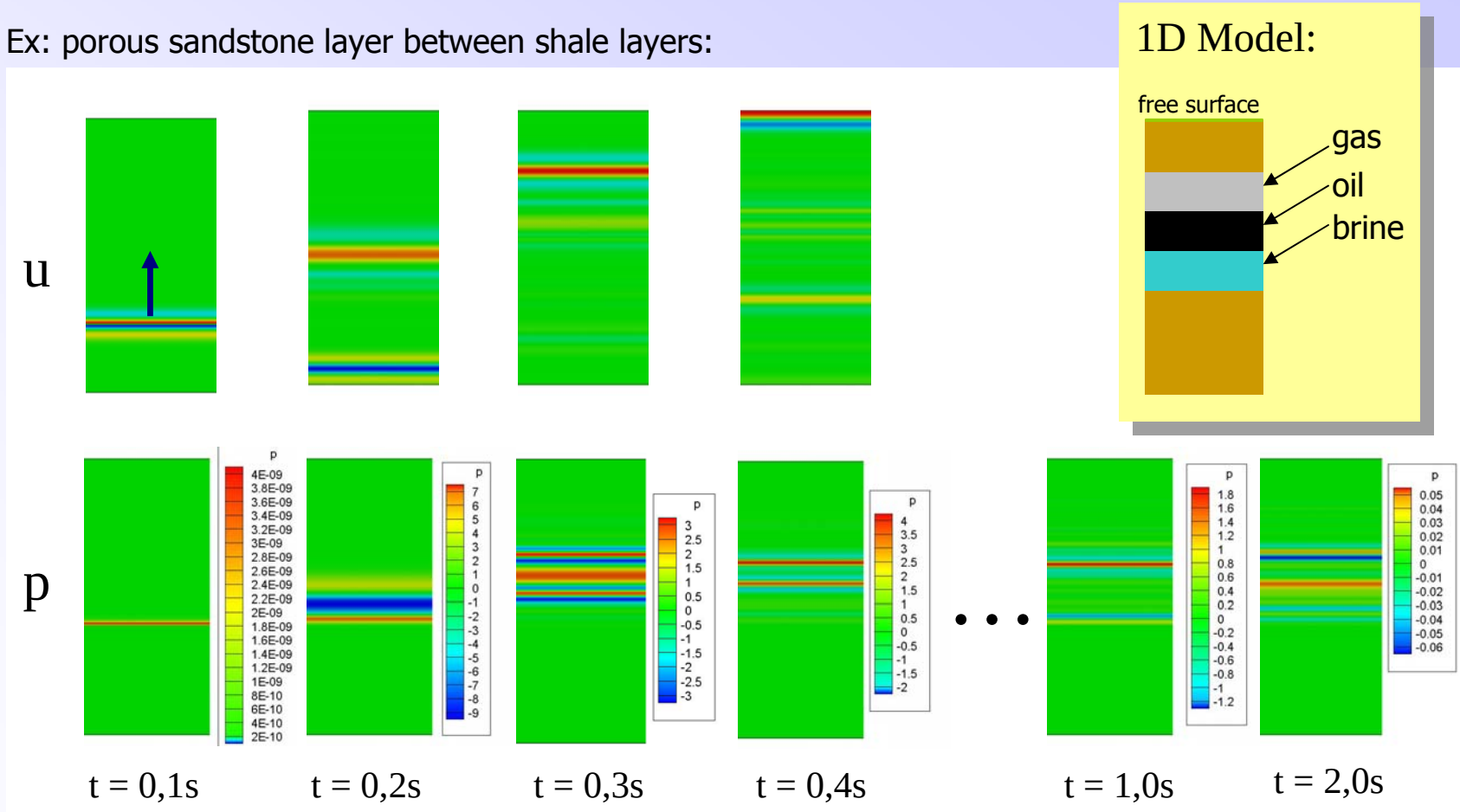
… and compare to an analytical solution based upon Carcione & Quiroga-Goode (1995) for a point explosion:

- Ricker wavelet 4.5Hz, homogenous viscous poroelastic material.
- Offset = 1000m.
- Fast wave: c=3836m/s; Q>300,000
- Slow wave: c=10m/s; Q=1
- Viscosity = 1cP => $f_c \approx$ 106KHz
- Very stiff problem!
- ADER-DG(ST) $O5$ simulation on 625,000 elements; 11 x 11 x 11Km; Δx~225m

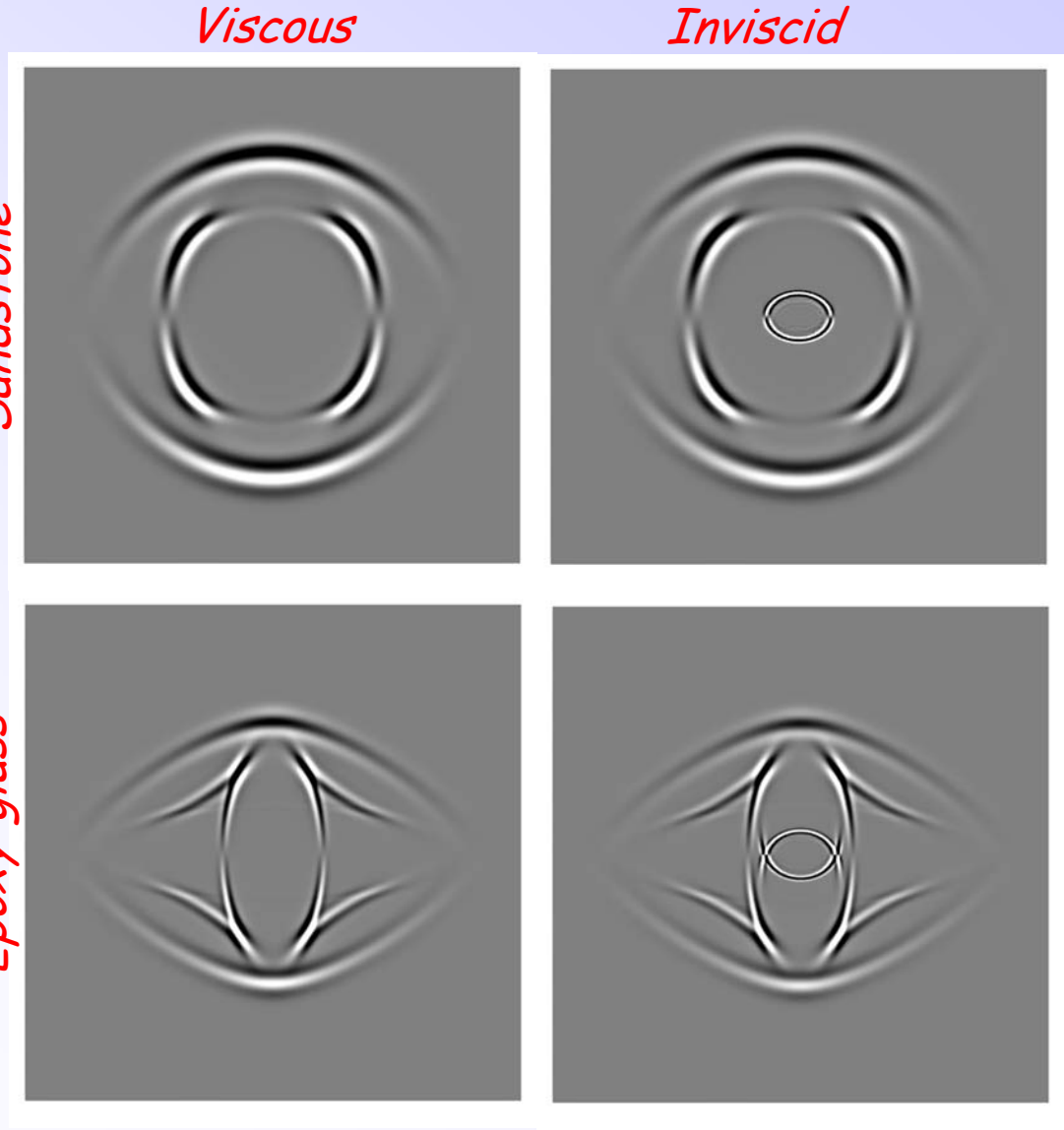
If it's not in the seismogram it doesn't matter… or does it?

The diffusive peak in the LF range does not only happen at the source location but also around material heterogeneities in the porous material. Particularly, elastic/poroelastic interfaces provoke larger diffusive fields.

Ex: porous sandstone layer between shale layers:



...and why not anisotropic too?



Transversely isotropic symmetry class. The saturating fluid is brine, point vertical source at around 3.5KHz. Domain is a 18x18m square

Examples taken from Carcione (1998) and recomputed with ADER-DG(ST) *O*4.

Anisotropy is in nature often much stronger in pore-related parameters, which strongly influence the behaviour of the slow wave.

	SANDSTONE	EPOXY
C_{11}	$71.8 \cdot 10^9$	$39.4 \cdot 10^9$
C_{12}	$1.2 \cdot 10^9$	$5.8 \cdot 10^9$
C_{13}	$3.2 \cdot 10^9$	$1.0 \cdot 10^9$
C_{22}	$53.4 \cdot 10^9$	$13.1 \cdot 10^9$
C_{44}	$26.1 \cdot 10^9$	$3.0 \cdot 10^9$
k_x	$600 \cdot 10^{15}$	$600 \cdot 10^{15}$
k_y	$100 \cdot 10^{15}$	$100 \cdot 10^{15}$
T_x	2	2
T_y	3.6	3.6

Conclusions

We have verified qualitatitively and quantitatively the performance of the ADER-DG method for poroelastic rheologies for ALL frequency ranges.

The anisotropy of the matrix has been included.

The method has still high-order convergence properties as time and space are discretized with the same polynomial degree.

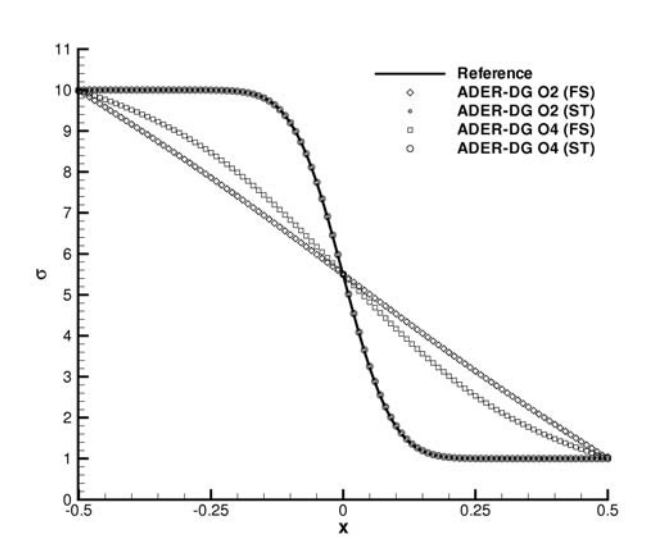
In every case we use the very flexible tetrahedral meshes as computational domain. Although not competitive for regular/cartesian problems, it is highly recomendable for realistic setups.

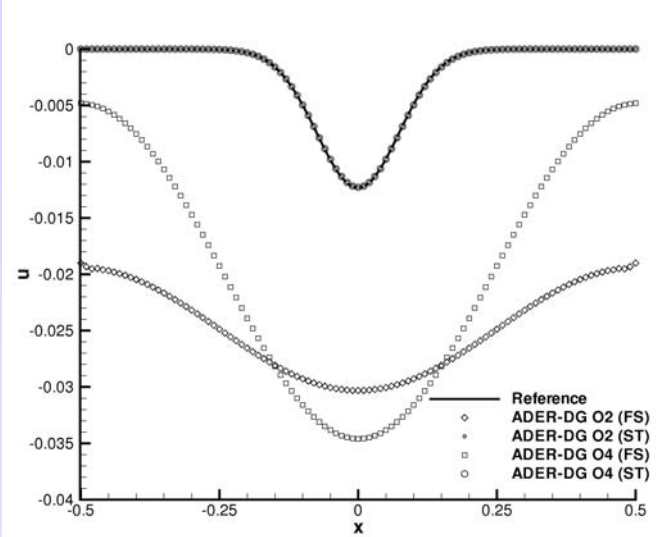
The scheme is parallelized and allows for local-timestepping an p-adaptivity.

The ADER-DG method offers accurate solutions for scenarios which are both **geometrically** and **rheologically** complex.

Assymptotical consistency:

Same 1D comparison, now 200 times longer simulation:





It can be proved that the poroelastic equations (hyperbolic) change their type to a mixed hyperbolic-parabolic type in the very stiff case. FS methods fail to capture such effects.

HYPERBOLIC

$$
\begin{aligned}
\frac{\partial}{\partial t}\sigma - c\frac{\partial}{\partial x}u - \alpha M\frac{\partial}{\partial x}v &= 0,\\
\frac{\partial}{\partial t}u - \frac{1}{\rho_1}\frac{\partial}{\partial x}\sigma - \frac{\beta_1}{\rho_1}\frac{\partial}{\partial x}p &= \frac{1}{\epsilon_1}v,\\
\frac{\partial}{\partial t}p + \alpha M\frac{\partial}{\partial x}u + M\frac{\partial}{\partial x}v &= 0,\\
\frac{\partial}{\partial t}v + \frac{1}{\rho_2}\frac{\partial}{\partial x}\sigma + \frac{\beta_2}{\rho_2}\frac{\partial}{\partial x}p &= -\frac{1}{\epsilon_2}v.
\end{aligned}
$$

$\varepsilon_2 \to 0$

HYPERBOLIC-PARABOLIC

$$
\begin{aligned}
\frac{\partial}{\partial t}\sigma - c\frac{\partial}{\partial x}u + \epsilon_2\alpha M\left(\frac{1}{\rho_2}\frac{\partial^2}{\partial x^2}\sigma + \frac{\beta_2}{\rho_2}\frac{\partial^2}{\partial x^2}p\right) &= 0,\\
\frac{\partial}{\partial t}u\left(\frac{\epsilon_2}{\epsilon_1}\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)\frac{\partial}{\partial x}\sigma + \left(\frac{\epsilon_2}{\epsilon_1}\frac{\beta_2}{\rho_2} - \frac{\beta_1}{\rho_1}\right)\frac{\partial}{\partial x}p &= 0,\\
\frac{\partial}{\partial t}p + \alpha M\frac{\partial}{\partial x}u + \epsilon_2 M\left(\frac{1}{\rho_2}\frac{\partial^2}{\partial x^2}\sigma + \frac{\beta_2}{\rho_2}\frac{\partial^2}{\partial x^2}p\right) &= 0.
\end{aligned}
$$

Poroelastic jacobians and PDE:

Biot's equations can be represented as a first-order PDE system in matrix form:

$$\frac{\partial Q_p}{\partial t} + A_{pq}\frac{\partial Q_q}{\partial x} + B_{pq}\frac{\partial Q_q}{\partial y} + C_{pq}\frac{\partial Q_q}{\partial z} = E_{pq}Q_q$$

With the vector of time-dependant variables:

$$Q = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{xz}, u, v, w, p, u_f, v_f, w_f)^T$$

The A, B, C (jacobians) and E matrices have time independent parameters of the material. Each tetrahedron has piecewise constant elements. The E matrix called "reaction term" is the only responsible for energy losses and, in this case, also for the stiff behaviour of the system at low frequencies.

$$A_{pq} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -c^u_{11} & -c^u_{16} & -c^u_{15} & 0 & -\alpha_1 M & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -c^u_{12} & -c^u_{26} & -c^u_{25} & 0 & -\alpha_2 M & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -c^u_{13} & -c^u_{36} & -c^u_{35} & 0 & -\alpha_3 M & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -c^u_{16} & -c^u_{66} & -c^u_{56} & 0 & -\alpha_6 M & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -c^u_{14} & -c^u_{46} & -c^u_{45} & 0 & -\alpha_4 M & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -c^u_{15} & -c^u_{56} & -c^u_{55} & 0 & -\alpha_5 M & 0 & 0 \\
-\frac{1}{\rho_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\beta_1}{\rho_1} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\rho_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{\rho_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 M & \alpha_6 M & \alpha_5 M & 0 & 0 & 0 & 0 \\
-\frac{1}{\rho_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\beta_2}{\rho_2} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\rho_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{\rho_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$E_{pq} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_1\nu}{\rho_1\kappa} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_1\nu}{\rho_1\kappa} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_1\nu}{\rho_1\kappa} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_2\nu}{\rho_2\kappa} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_2\nu}{\rho_2\kappa} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_2\nu}{\rho_2\kappa}
\end{pmatrix}$$

The jacobians B and C are analogous to A with rearraged entries. The eigenvalues of A are exactly the wave velocities allowed for all wave modes allowed in the material. Its eigenvectors describe their associated plane waves.

The Space-Time DG method: