Traction-at-Split-Node Method for Dynamic Rupture Propagation: Numerical Comparison of the Finite-Difference and Finite-Element Implementations

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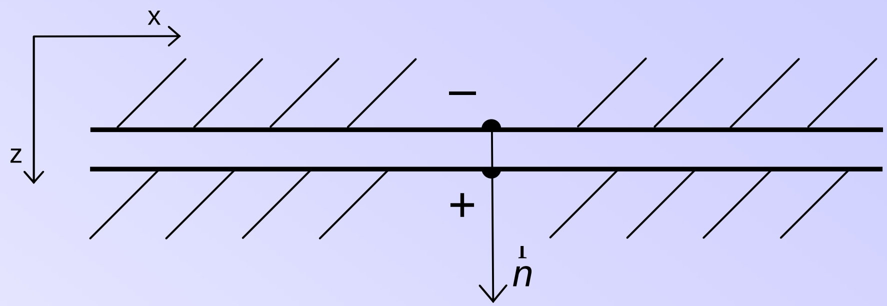
Outline

1. TSN Implementations
in the 4^{th}-order staggered-grid
VS FD scheme

2. TSN Implementation
in the FEM

3. Numerical comparison
of the FD and FE implementations

Basic definitions



Slip: $\Delta\vec{u}(\vec{x},t) = \vec{u}^{+}(\vec{x}^{+},t) - \vec{u}^{-}(\vec{x}^{-},t)$

Slip-rate: $\Delta\vec{v}(\vec{x},t) = \vec{v}^{+}(\vec{x}^{+},t) - \vec{v}^{-}(\vec{x}^{-},t)$

Total traction: $\vec{T}(\vec{n};\vec{x},t) = \vec{T}^{0}(\vec{n};\vec{x}) + \Delta\vec{T}(\vec{n};\vec{x},t)$

$\vec{T}^{0}(\vec{n};\vec{x})$
- initial traction
- due to tectonic load
- constant in time
- does not cause motion

$\Delta\vec{T}(\vec{n};\vec{x},t)$
- traction variation due to rupture and wave propagation

Boundary conditions

$$\left|\vec{T}_{sh}^{ct}\right| \le S \quad \Rightarrow \quad \text{No Rupture}$$

$$\vec{T}_n^{c} = \vec{T}_n^{ct}$$

$$\vec{T}_{sh}^{c} = \vec{T}_{sh}^{ct}$$

$$D\vec{v}_{sh}\left(t + \tfrac{dt}{2}\right) = 0$$

Trial traction $\vec{T}^{ct}$ keeps partial nodes to move together
(as if they were glued)

$$\left|\vec{T}_{sh}^{ct}\right| > S \quad \Rightarrow \quad \text{Rupture}$$

$$\vec{T}_n^{c} = \vec{T}_n^{ct}$$

$$\vec{T}_{sh}^{c} = S \frac{\vec{T}_{sh}^{ct}}{\left|\vec{T}_{sh}^{ct}\right|}$$

$$D\vec{v}_{sh}\left(t + \tfrac{dt}{2}\right) = dt\, B \left[\vec{T}_{sh}^{ct} - \vec{T}_{sh}^{c}\right]$$

TSN implementations in the FD schemes

Andrews (1973, 1999)

2nd-order FD scheme
with spatial differentiation equivalent to FEM
? 1st-order at the fault surface

Day (1977, 1982)

2nd-order FD scheme
on a partly-staggered grid
? 1st-order at the fault surface

Dalguer and Day (2007)

4th-order staggered-grid velocity-stress FD scheme
2nd-order close to the fault surface
1st-order at the fault surface

Our TSN implementations into FD schemes

We presented
3 different TSN implementations in the FD scheme
in Moczo et at. (2007)

FD scheme

4^{th}-order staggered-grid velocity-stress FD scheme

Approximations near and at the fault surface in the TSN implementations

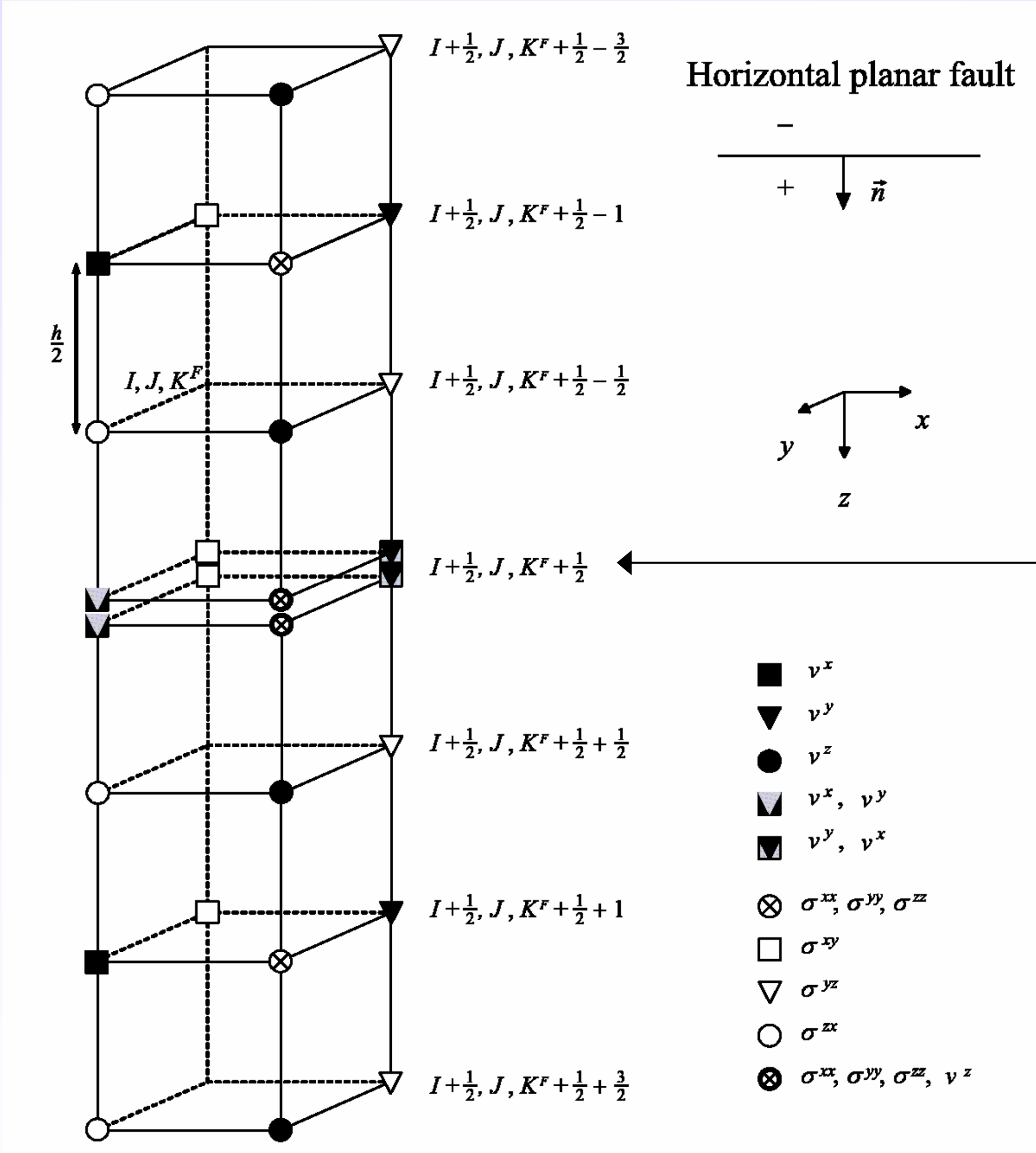
2^{nd}-order

4^{th}-order

mixed 2^{nd} – 4^{th}-order

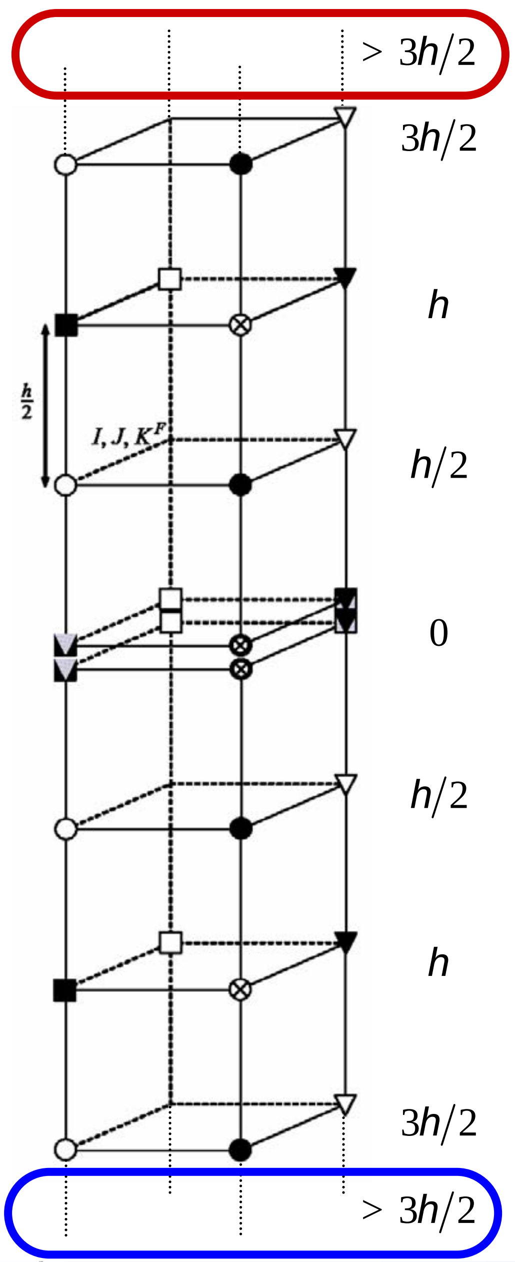
Moczo, P., J. Kristek, M. Galis, P. Pazak a M. Balazovjech, 2007.
The Finite-Difference and Finite-Element Modeling of Seismic Wave Propagation and Earthquake Motion.
Acta Physica Slovaca, **57**, No. 2, 177 - 406.

SG FD 2nd-order TSN implementation



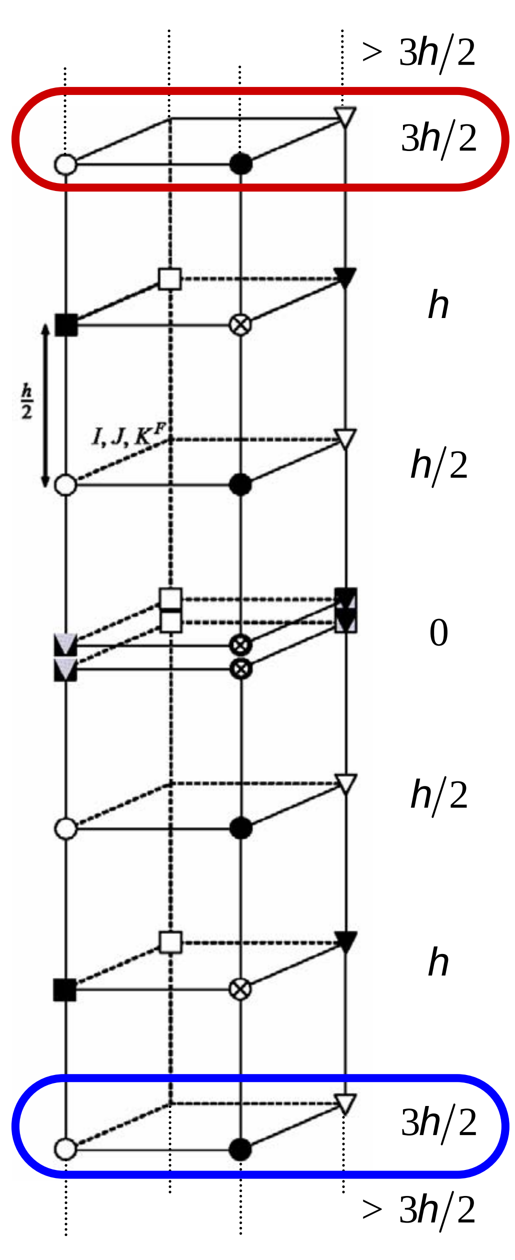
Fault plane,
that is,
grid plane
covered by split nodes

SG FD 2nd-order TSN implementation



Derivatives
in the ***x***-, ***y***-, and ***z***- directions
are approximated using
standard 4th-order FD formulas

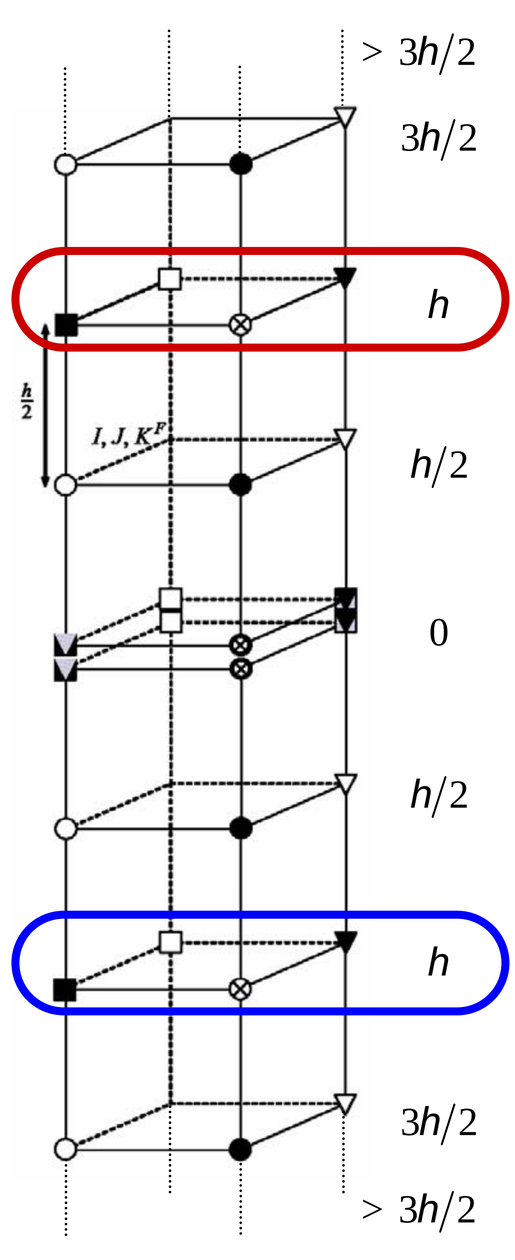
SG FD 2^{nd}-order TSN implementation



Derivatives
in the ***x***-, and ***y***-directions
are approximated using
standard 4^{th}-order FD formulas

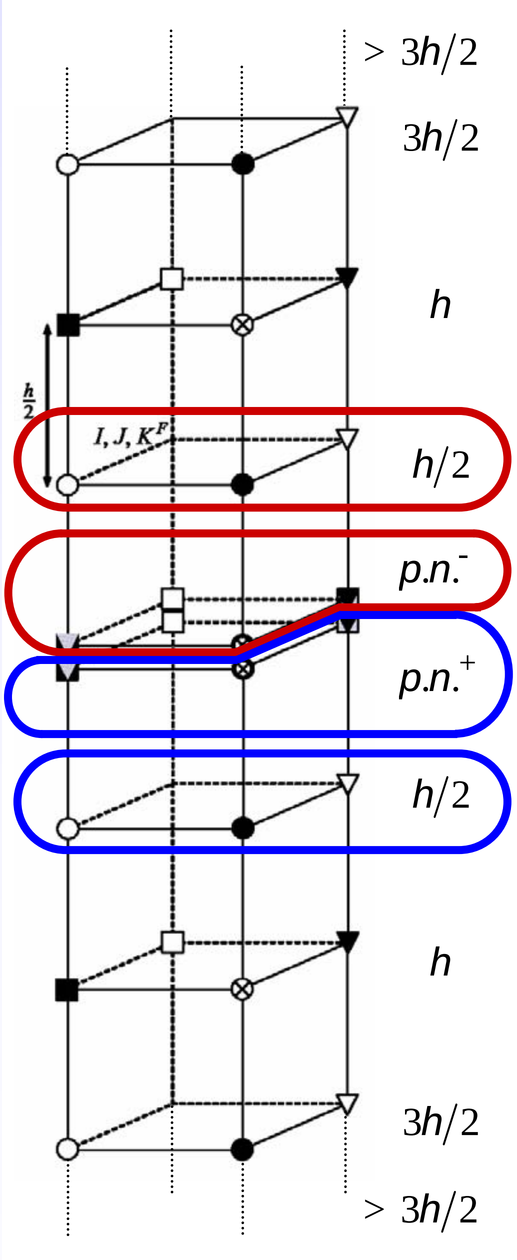
Derivatives
in the ***z***-direction
are approximated using
the **adjusted 4^{th}-order** FD formulas

SG FD 2^{nd}-order TSN implementation



Derivatives
in the ***x***-, ***y***-, and ***z***- directions
are approximated using
standard 2^{nd}-order FD formulas

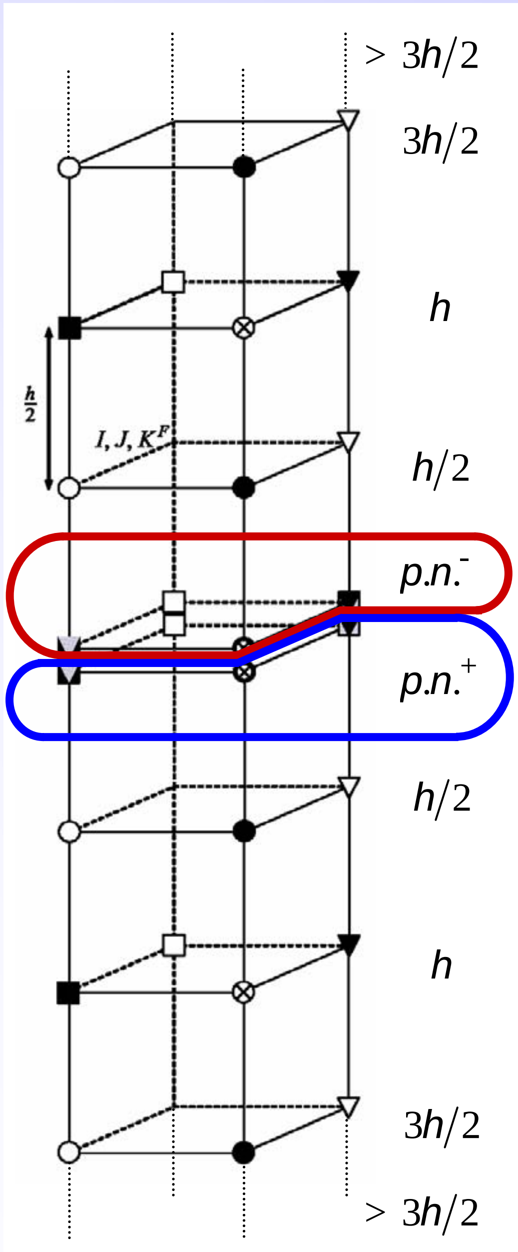
SG FD 2nd-order TSN implementation



Derivatives
in the ***x***-, and ***y***-directions
are approximated using
standard 2nd-order FD formulas

Derivatives
in the ***z***-direction
are approximated using
adjusted 2nd-order FD formulas

SG FD 2nd-order TSN implementation



Requirement

$$Dv^{\gamma,m+\frac{1}{2}} = 0$$

leads to trial traction

$$T^{ct,\gamma,m} = T^{0,\gamma} + \frac{3}{8}h\left[\frac{\frac{1}{\Delta t}\rho^{-}\rho^{+}Dv^{\gamma,m-\frac{1}{2}} + \rho^{-}F^{+,\gamma,m} - \rho^{+}F^{-,\gamma,m}}{\rho^{-}+\rho^{+}}\right]$$

$$\gamma \in \{x,y\}$$

$$F^{\mp,\gamma,m} = D_x^{(2)}\sigma^{\mp,x\gamma,m} + D_y^{(2)}\sigma^{\mp,y\gamma,m} + \left(\partial_z\sigma^{z\gamma}\right)^{\mp,m}$$

$$\left(\partial_z\sigma^{\gamma z}\right)^{-} \approx \frac{1}{h}\left\{0 - 3\,\sigma^{\gamma z}\left(-\tfrac{h}{2}\right) + \tfrac{1}{3}\sigma^{\gamma z}\left(-\tfrac{3h}{2}\right)\right\}$$

$$\left(\partial_z\sigma^{\gamma z}\right)^{+} \approx \frac{1}{h}\left\{0 + 3\,\sigma^{\gamma z}\left(+\tfrac{h}{2}\right) - \tfrac{1}{3}\sigma^{\gamma z}\left(+\tfrac{3h}{2}\right)\right\}$$

SG FD 4^{th}-order and SG FD mixed 2^{nd} – 4^{th}-order TSN implementations

SG FD 4^{th}-order

Analogous to the SG FD 2^{nd}-order
but using the 4^{th}-order standard
and 4^{th}-order adjusted FD approximations

SG FD mixed 2^{nd} – 4^{th}-order

Combination of
2^{nd}-order and 4^{th}-order approximation
near and at the fault surface

FEM implementation of TSN

The trial traction for FEM has a form

$$\vec{T}^{\,ct,m} = \vec{T}^{\,0} + \frac{\frac{1}{\Delta t} M^{-} M^{+} \Delta \vec{v}^{\,m-\frac{1}{2}} + M^{-} \vec{F}^{+,m} - M^{+} \vec{F}^{-,m}}{A\left(M^{-} + M^{+}\right)}$$

where forces $\vec{F}^{+,m}$ and $\vec{F}^{-,m}$ are

the components of the global restoring force vector

The requirement of the TSN

The force acting at one node
can affect only that node during one time level

To satisfy this condition
we have to use diagonal mass matrix

If mass matrix is not diagonal,
we have to use lumped mass matrix

Numerical tests - Configuration

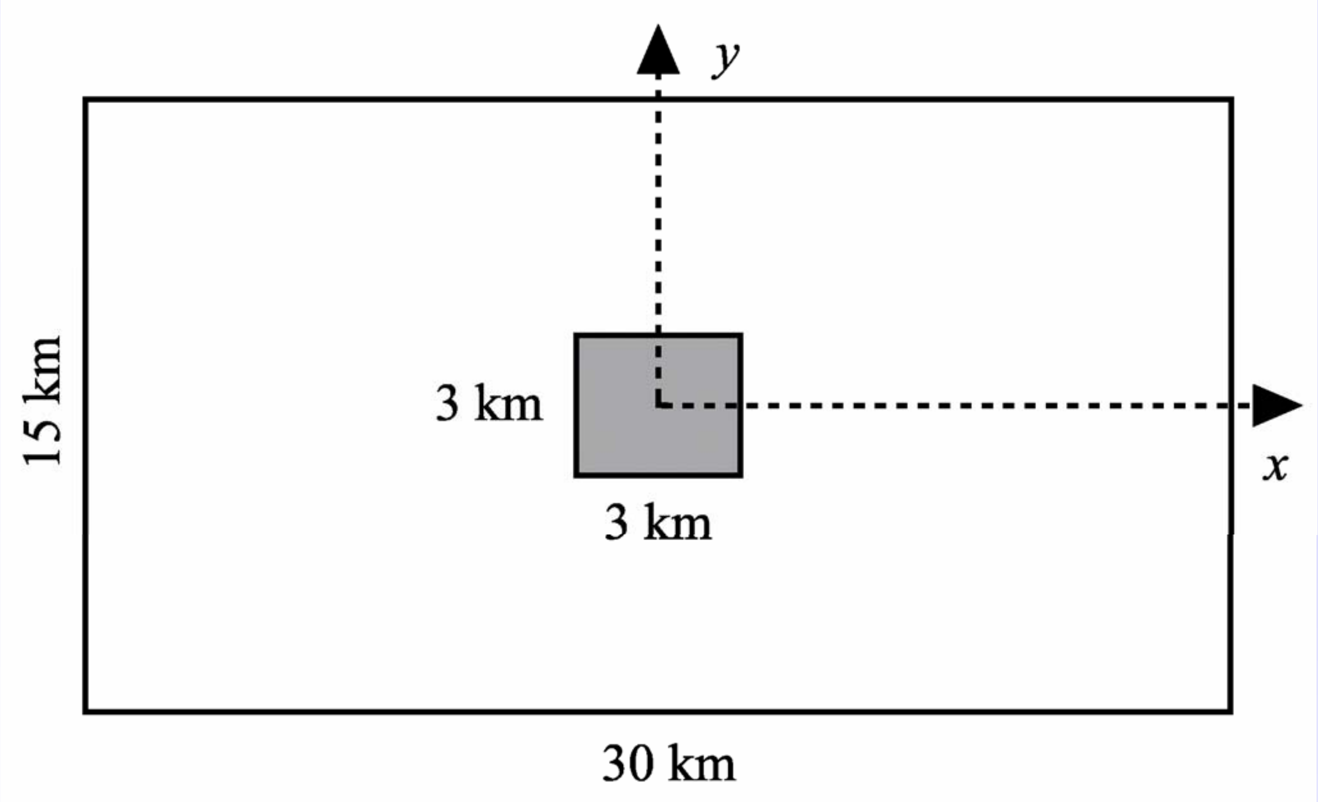
Day et al. (2005)
compared
TSN - FD scheme on partly staggered grid
and
Boundary integral method

Dalguer and Day (2006)
compared
TSN - FD scheme on a partly staggered grid,
Thick-fault method
and
Stress-glut method

Dalguer and Day (2007)
compared
TSN - VS FD scheme on staggered grid
and
TSN - FD scheme on partly staggered grid

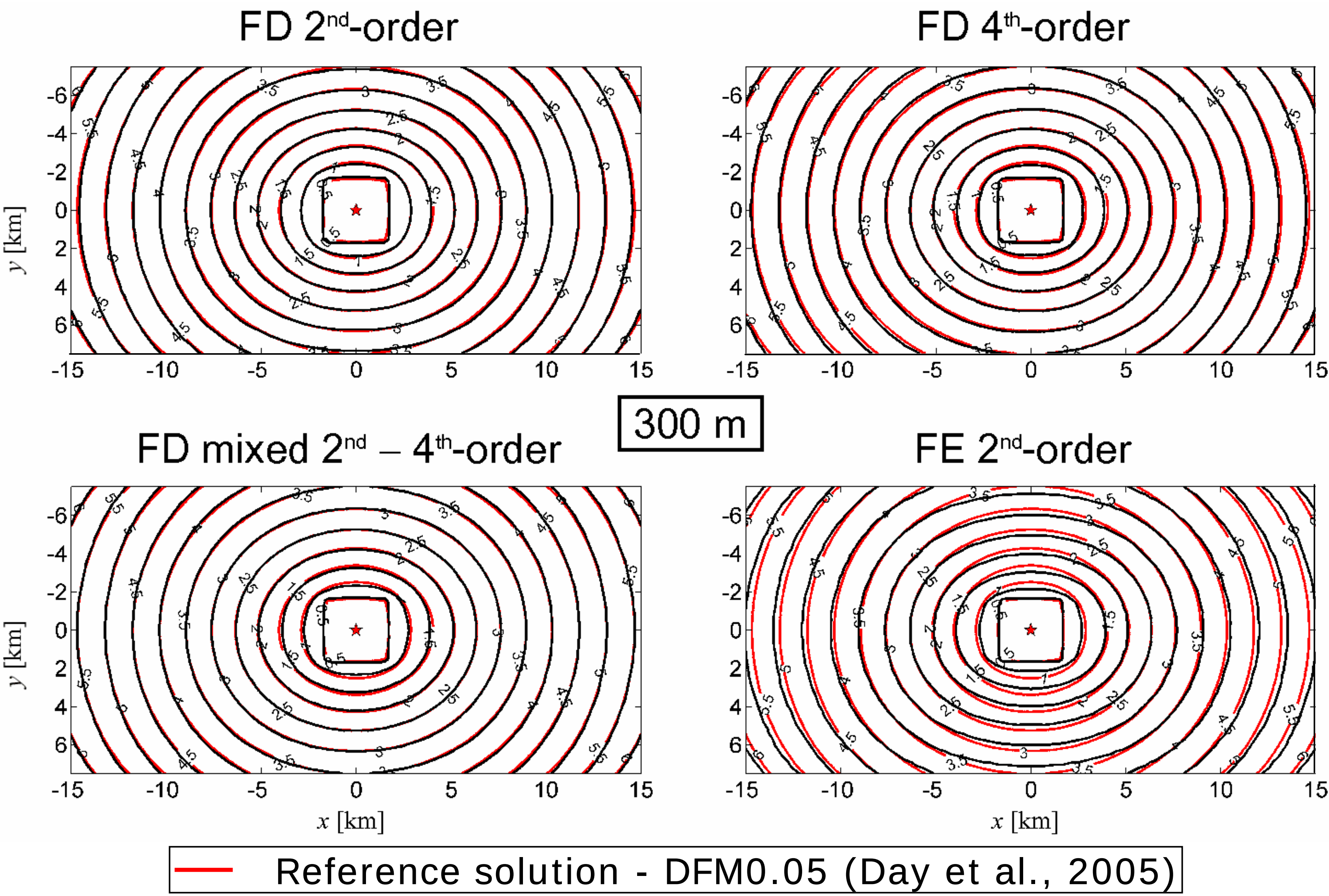
Numerical tests - Configuration

The test configuration
is based
on the TPV3
of the SCEC Dynamic Rupture Benchmark



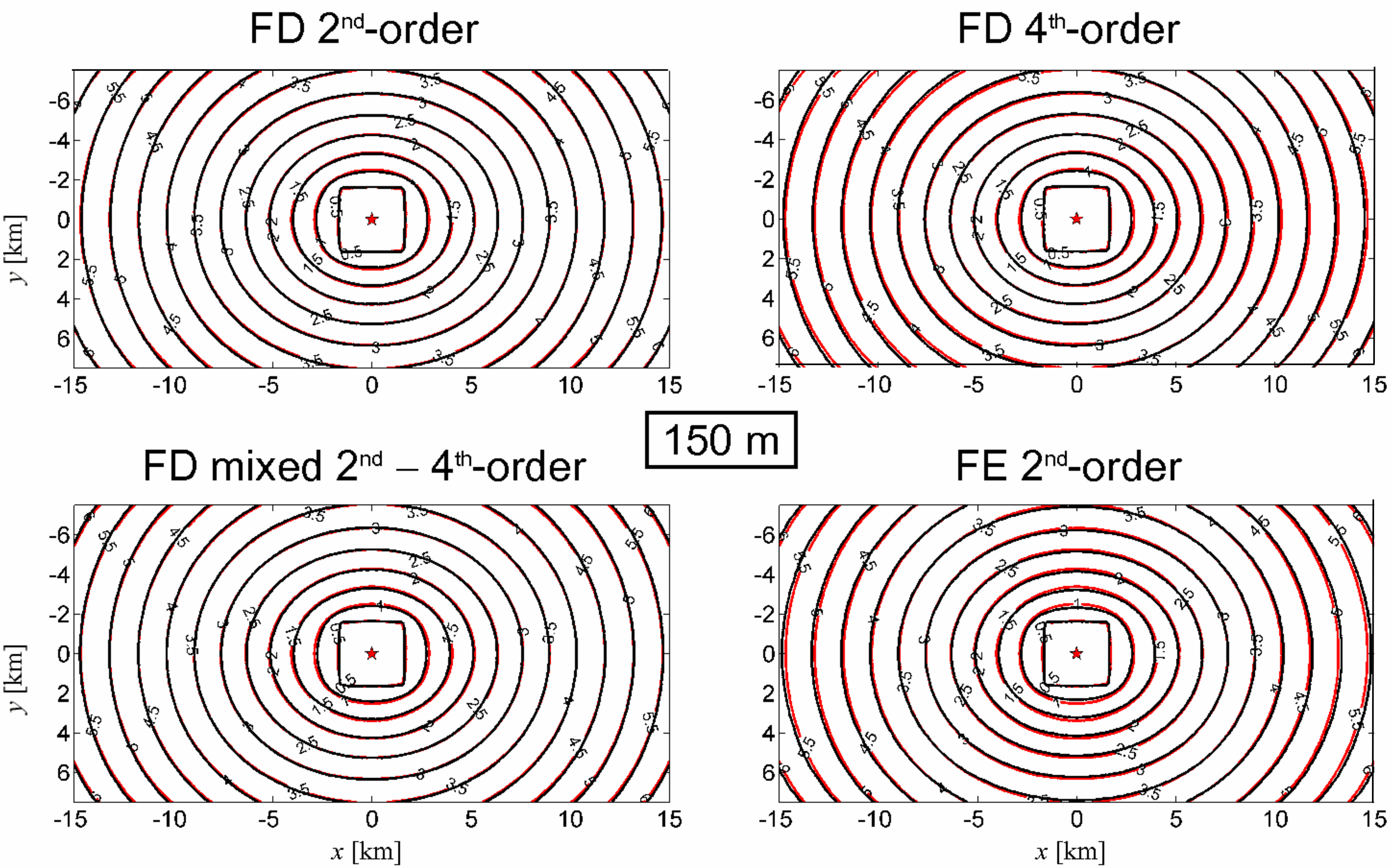
Numerical test - Results

Rupture propagation time



Numerical test - Results

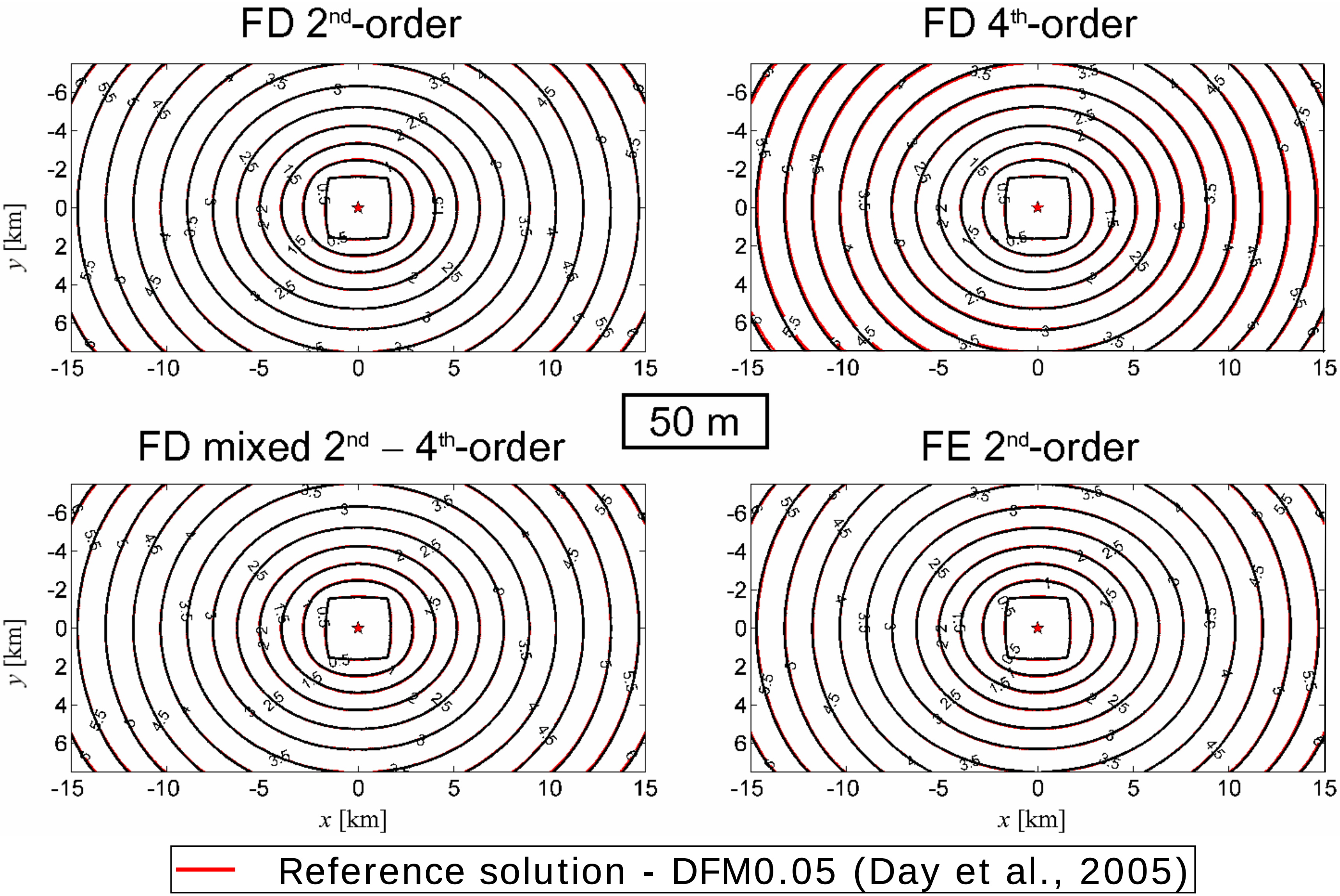
Rupture propagation time



Reference solution - DFM0.05 (Day et al., 2005)

Numerical test - Results

Rupture propagation time



Numerical test - Results

1) At each point of the fault
we compute
RMS of the difference
between
the rupture propagation times

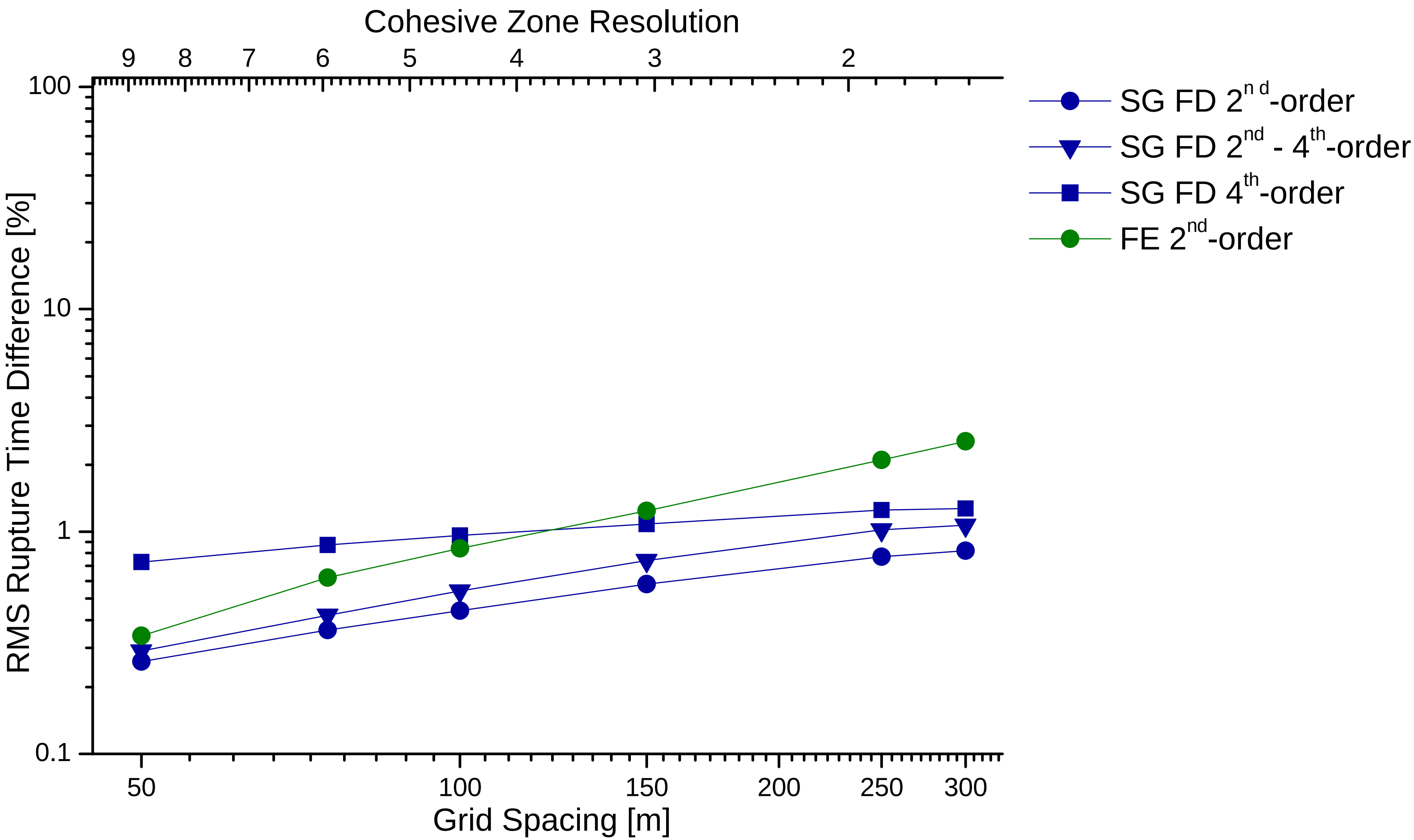
in the tested solution
and
in the reference solution

2) Find **the max. value**
of RMS rupture time difference
at the fault plane

3) Find **the average value**
of RMS rupture time difference
at the fault plane

Numerical test - Results

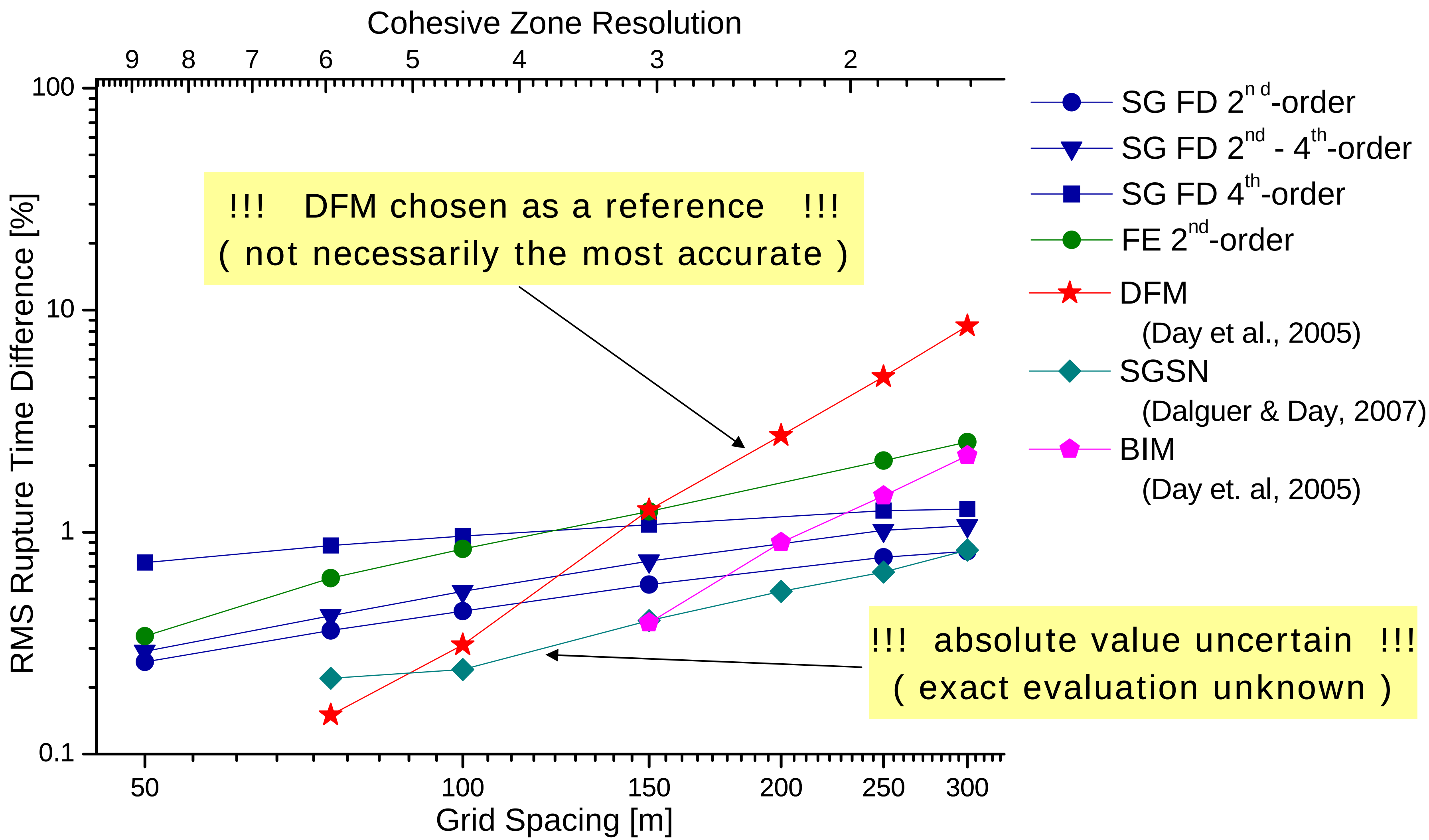
Relative convergence rate



RMS rupture time difference evaluated relative to DFM0.05 of Day et al. (2005)

Numerical test - Results

Relative convergence rate

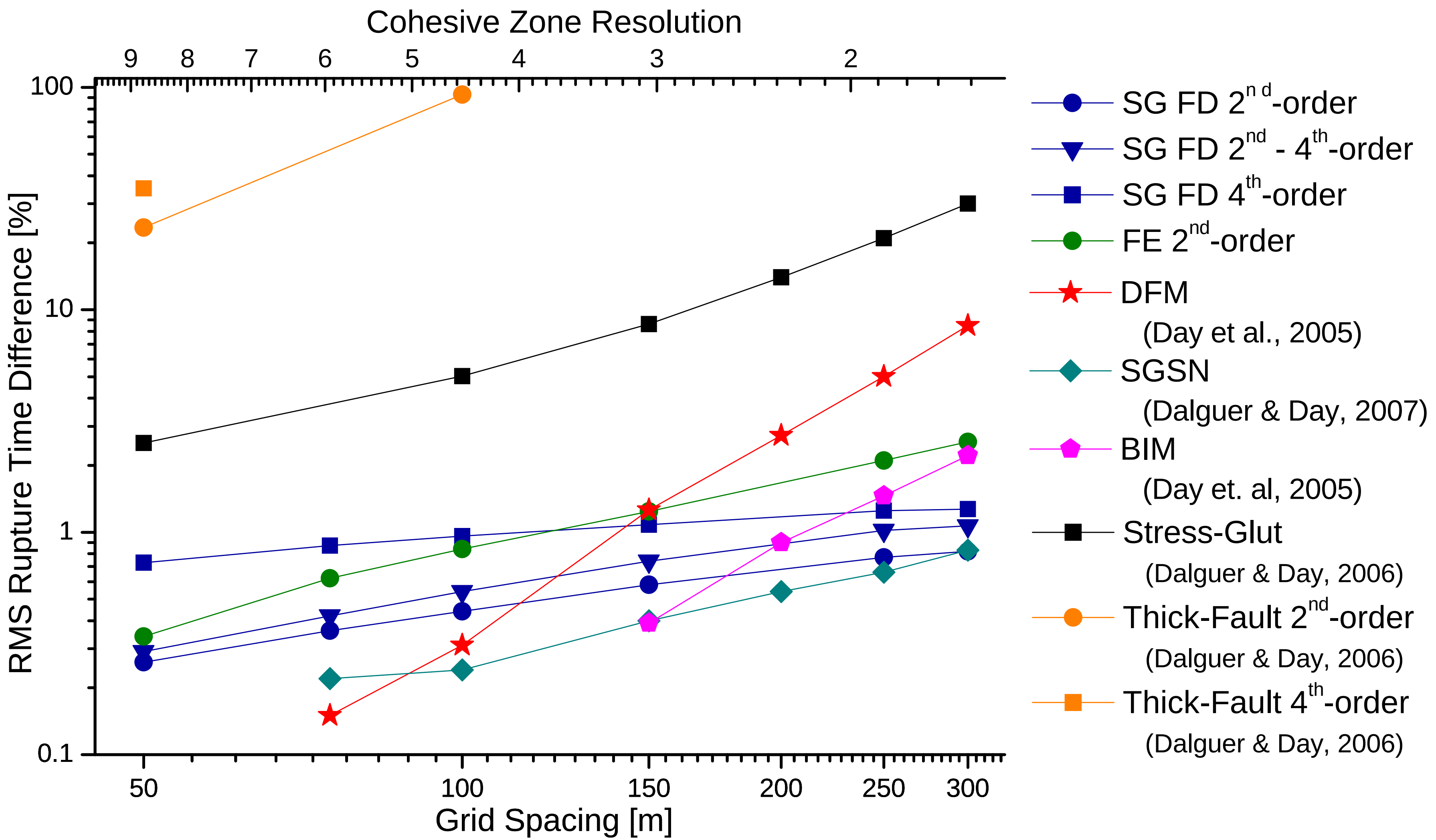


BIM: RMS rupture time difference evaluated relative to BI0.1 of Day et al. (2005)

Other: RMS rupture time difference evaluated relative to DFM0.05 of Day et al. (2005)

Numerical test - Results

Relative convergence rate

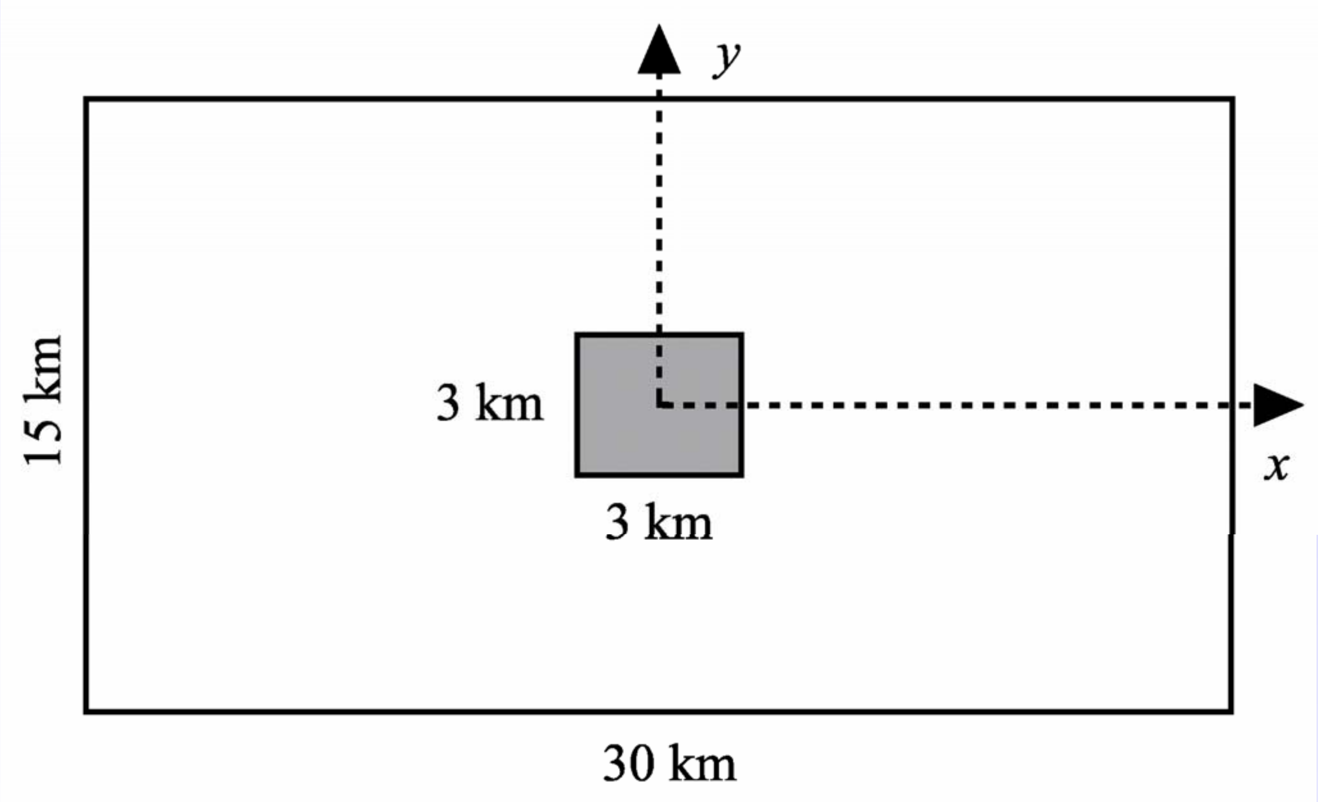


BIM: RMS rupture time difference evaluated relative to BI0.1 of Day et al. (2005)

Other: RMS rupture time difference evaluated relative to DFM0.05 of Day et al. (2005)

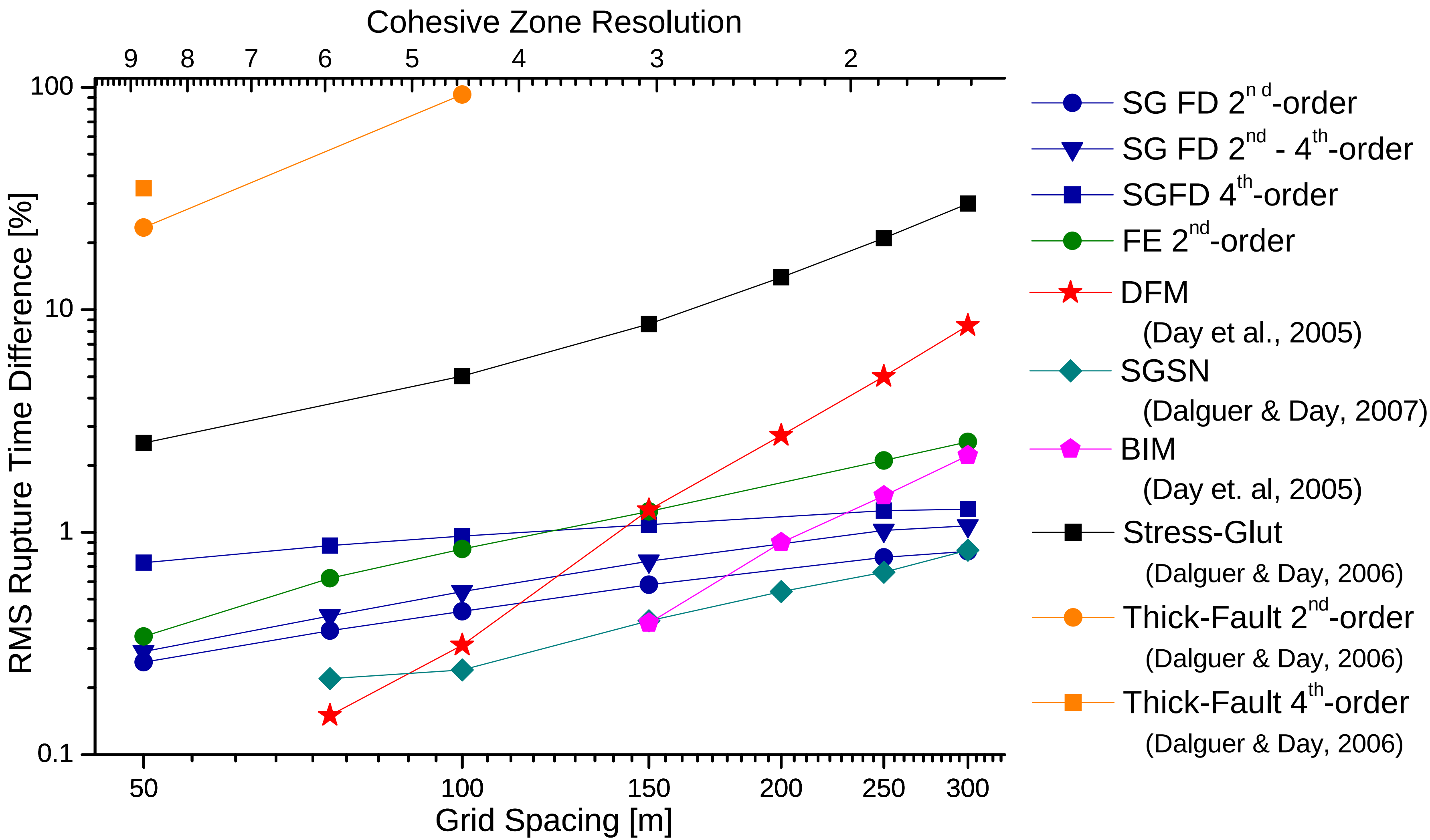
The test configuration is ill-posed! Why?

1) Discontinuous traction vector is prescribed at the edge/boundary of the nucleation zone



Numerical test - Results

Relative convergence rate



BIM: RMS rupture time difference evaluated relative to BI0.1 of Day et al. (2005)

Other: RMS rupture time difference evaluated relative to DFM0.05 of Day et al. (2005)

Conclusions

We implemented 3 TSN algorithms
in the 4th-order staggered-grid VS FD scheme:

FD SG 2nd-order,
FD SG 4th-order and
FD SG mixed 2nd – 4th-order

We compared
our 3 FD TSN implementations
with
our FE TSN implementation
following Day et al. (2005), Dalguer a Day (2006,2007)

Comparison showed that
the relative convergence rate of the FD SG 4th-order is lowest
and
the relative convergence rate of the FE 2nd-order is highest

Outlook

We want to use the experience
with this comparison
to define an improved model configuration
for the SPICE Code Validation

Thank you

for your attention